# Lyapunov functions for switched linear systems: Proof of convergence for an LP computational approach

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***Abstract*— A recent approach uses linear programming (LP) to compute continuous and piecewise affine (CPA) Lyapunov functions for arbitrary switched linear systems. Such a Lyapunov function is a common Lyapunov function (CLF) for all the respective linear subsystems and asserts the exponential stability of the equilibrium at the origin for the switched system. In this letter, we prove that this LP approach is constructive, i.e., that it succeeds in computing a Lyapunov function for the switched system, whenever the origin is exponentially stable.**

***Index Terms*— Common Lyapunov function, Linear Programming, Linear systems, Switched Systems.**

## I. Introduction

In [1] a linear programming (LP) approach to compute a common Lyapunov function (CLF) for a finite set of linear systems

$$\dot{\mathbf{x}} = A_i\mathbf{x}, \quad A_i \in \mathbb{R}^{n\times n}, \quad i = 1, 2, \ldots, N, \tag{1}$$

was presented. It is well known, that the existence of such a CLF is equivalent to the exponential stability of the equilibrium at the origin for the corresponding arbitrary switched system $\dot{\mathbf{x}} \in \mathrm{co}\{A_i\mathbf{x}\}$; cf. e.g. [7], [14], [22], [23] for some general references for switched systems and stability. We talk about a CLF for the systems (1) and a Lyapunov function for the (arbitrary) switched system (1) interchangeably.

The origin is an exponentially stable equilibrium for a linear system $\dot{\mathbf{x}} = A\mathbf{x}$, if and only if it possesses a quadratic Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$. This is equivalent to the existence of a symmetric and positive definite matrix $P$ such that $P^T A + AP$ is negative definite. Therefore, if the stability of the origin is to be investigated for (1), it is most natural to search for a quadratic common Lyapunov function (QCLF). This can be done by solving a linear matrix inequality (LMI): Find a symmetric and positive definite $P \in \mathbb{R}^{n\times n}$ such that $A_i^T P + PA_i$ is negative definite for all $i$; in formula:

$$P \succ 0 \text{ and } A_i^T P + PA_i \prec 0 \text{ for } i = 1, 2, \ldots, N. \tag{2}$$

The limitation of this LMI approach is that the origin might be exponentially stable for (1), and thus there exist CLFs for the subsystems, but none of these CLFs is quadratic.

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Hence, numerous methods computing norms (Minkowskii, weighted), that can serve as Lyapunov functions for switched linear systems, have surfaced in the literature. They usually use linear programming (LP) to parameterize a Lyapunov function, see e.g. [4]–[6], [10], [20], [21], [24], [25]; the converse theorems from [18], [19] have been important for many of these approaches.

In [1] yet another LP approach to compute a CLF for (1) was presented, that is an adaptation of the so-called CPA method to compute Lyapunov functions, cf. e.g. [3], [9], [13], [15], to arbitrarily switched linear systems. While the examples in [1] suggested that this method is more general than the LMI approach (2), a proof of convergence was not delivered. The main contribution of this letter is to deliver such a proof in Theorem 2.

Suitable classes of Lyapunov functions for linear switched systems were recently studied in [17], see also [12], [16], where it was shown that the class of piecewise linear functions is large enough when searching for CLFs for (1), i.e. there always exists a piecewise linear Lyapunov function if the origin is exponentially stable. These results, however, are not directly applicable to prove that the LP approach from [1] is always able to compute a Lyapunov function when the origin is exponentially stable, because it uses an a priori fixed triangulation. However, in Section 2 in [16] a converse theorem is delivered that allows one to prove that it is indeed constructive.

In Section II we recall some basic facts about triangulations necessary for the CPA method and the LP approach from [1], before we prove our main results in Section III, that the LP approach is constructive, i.e., that it succeeds in computing a Lyapunov function for the arbitrary switched system (1), whenever the origin is exponentially stable.

*Remark: The case $n = 1$ is trivial for the discussion in this letter. Therefore we assume that $n \geq 2$ in the whole letter.*

### A. Notation

We write vectors $\mathbf{x} \in \mathbb{R}^n$ in bold face and assume they are column vectors. For a vector $\mathbf{x}$ we write $x_i$ for its $i$th component. For a matrix $A$ and vector $\mathbf{x}$ we write $A^{\mathrm{T}}$ and $\mathbf{x}^{\mathrm{T}}$ for their transposes, respectively. For vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m \in \mathbb{R}^n$ we write $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ for the $\mathbb{R}^{n\times m}$ matrix with $\mathbf{x}_i$ in its $i$th column.

For vectors $\mathbf{x} \in \mathbb{R}^n$ and $p \geq 1$ we define their $p$-norms through $\|\mathbf{x}\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$. For $p = \infty$ we define

$\|\mathbf{x}\|_\infty := \lim_{p\to\infty}\|\mathbf{x}\|_p = \max_{i\in\{1,2,\dots,n\}}|x_i|$. Recall the norm equivalence relation ($1/\infty := 0$ and $n^0 := 1$)

$$\|\mathbf{x}\|_p \le \|\mathbf{x}\|_q \le n^{q^{-1}-p^{-1}}\|\mathbf{x}\|_p \quad \text{for } \infty \ge p > q \ge 1.$$

We define $\mathrm{dist}(A,B) := \inf_{\mathbf{x}\in A,\mathbf{y}\in B}\|\mathbf{x}-\mathbf{y}\|_2$ for nonempty $A, B \subset \mathbb{R}^n$.

A (vector) norm $\|\cdot\|$ on $\mathbb{R}^n$ induces a (matrix) norm on $\mathbb{R}^{n\times n}$ through $\|A\| := \max_{\|\mathbf{x}\|=1}\|A\mathbf{x}\|$. An induced matrix norm is sub-multiplicative, i.e. $\|AB\| \le \|A\|\,\|B\|$ for all $A, B \in \mathbb{R}^{n\times n}$, and obviously $\|A\mathbf{x}\| \le \|A\|\|\mathbf{x}\|$ for $A \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$.

With $p = 1$, $p = 2$, and $p = \infty$ there are simple formulas for the induced matrix norms: $\|A\|_1 = \|A^{\mathrm{T}}\|_\infty = \max_i \|\mathbf{a}_i\|_1$, where $\mathbf{a}_i$ are the column vectors of $A$, and $\|A\|_2 = \max_{\|\mathbf{x}\|_2=1}\sqrt{\mathbf{x}^{\mathrm{T}}A^{\mathrm{T}}A\mathbf{x}}$ is the square root of the largest eigenvalue of the symmetric and positive-semidefinite matrix $A^{\mathrm{T}}A$. We denote by $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ the standard orthonormal basis of $\mathbb{R}^n$.

By $\mathrm{Sym}(C)$ we denote the set of the permutations of a set $C$, i.e. the set of bijective mappings $C \to C$. Recall that the gradient $\nabla V(\mathbf{x})$ of a function $V: \mathbb{R}^n \to \mathbb{R}$ at a point $\mathbf{x} \in \mathbb{R}^n$ is a row vector and therefore $\nabla V(\mathbf{x})\mathbf{y} \in \mathbb{R}$ is the scalar-product of the column vectors $[\nabla V(\mathbf{x})]^{\mathrm{T}}, \mathbf{y} \in \mathbb{R}^n$. Finally, $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ := \mathbb{N}_0 \setminus \{0\}$, and recall that a function $V: \mathbb{R}^n \to \mathbb{R}$ is said to be positively homogenous, if for every $c > 0$ and all $\mathbf{x} \in \mathbb{R}^n$ we have $V(c\mathbf{x}) = cV(\mathbf{x})$.

## II. Preliminaries

We recall some know results and definitions that we will use in the proof of Theorem 1, which in turn is used to prove the main results of this letter in Theorem 2. The sole purpose of this section is to fix the notation and enhance the readability of the proofs in Section III. First, in Section II-A we recall some definitions regarding triangulations as needed for the LP approach in [1]. Second, in Section II-B we state the LP approach from [1] to compute a CLF for the systems (1). Third, in Section II-C we recall the definitions of some concrete triangulation used in the LP approach in [1] and the proofs in Section III.

### A. Triangulations

The approach in [1] attempts to parameterize a continuous and piecewise affine (CPA) Lyapunov function using LP on a compact domain $\mathcal{D} \subset \mathbb{R}^n$ of the state-space of the system in question. Thus, first a triangulation $\mathcal{T}$ of the domain $\mathcal{D}$ is needed, i.e. a subdivision of $\mathcal{D}$ into simplices. An $n$-simplex $\mathfrak{S}_\nu \subset \mathbb{R}^n$ with vertices $\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \dots, \mathbf{x}_n^\nu \in \mathbb{R}^n$ is defined as

$$\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \dots, \mathbf{x}_n^\nu) := \left\{\sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu : \sum_{i=0}^n \lambda_i = 1 \text{ and all } \lambda_i \ge 0\right\}.$$

We write $\mathcal{D}_\mathcal{T}$ for the set-theoretic union of the simplices in $\mathcal{T}$ and say that $\mathcal{T}$ triangulates $\mathcal{D}_\mathcal{T} = \mathcal{D} \subset \mathbb{R}^n$. The triangulation must be shape-regular in the sense that two different simplices

$$\mathfrak{S}_\gamma := \mathrm{co}(\mathbf{x}_0^\gamma, \mathbf{x}_1^\gamma, \dots, \mathbf{x}_n^\gamma), \quad \gamma \in \{\nu, \mu\},$$

of the triangulation intersect in a common face

$$\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \mathrm{co}(\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_k), \quad \mathbf{y}_j = \mathbf{x}^\nu_{\ell_j^\nu} = \mathbf{x}^\mu_{\ell_j^\mu},$$

where $j = 0, 1, \dots, k < n$, $\ell_j^\nu, \ell_j^\mu \in \{0, 1, \dots, n\}$, and $\ell_j^\gamma \ne \ell_m^\gamma$ if $j \ne m$, $\gamma \in \{\nu, \mu\}$. We are only interested in non-degenerated simplices, i.e. $\mathfrak{S}_\nu \in \mathcal{T}$ has an $n$-dimensional volume strictly larger than zero or equivalently, the vertices $\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \dots, \mathbf{x}_n^\nu$ are affinely independent; see [8] for details. For our application, it is also convenient to assume that the vertices of a simplex $\mathfrak{S}_\nu \in \mathcal{T}$ have a fixed order and we write $\mathfrak{S}_\gamma = \mathrm{co}(\mathbf{x}_0^\gamma, \mathbf{x}_1^\gamma, \dots, \mathbf{x}_n^\gamma)$ rather than $\mathrm{co}\{\mathbf{x}_0^\gamma, \mathbf{x}_1^\gamma, \dots, \mathbf{x}_n^\gamma\}$, i.e. ordered tuple rather than a set. Note that the convex combination of the vectors is a subset of $\mathbb{R}^n$ and does not depend on their order, however, the matrix $X_\nu$ defined below does depend on the order.

An additional requirement of a triangulation for the LP problem from [1] is that all simplicies in the triangulation have the origin as a vertex; for simplicity we assume $\mathbf{x}_0^\nu = \mathbf{0}$ for all simplices $\mathfrak{S}_\nu \in \mathcal{T}$. For a simplex $\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \dots, \mathbf{x}_n^\nu)$ with $\mathbf{x}_0^\nu = \mathbf{0}$, we repeatedly use the matrix

$$X_\nu := (\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \dots, \mathbf{x}_n^\nu) \in \mathbb{R}^{n\times n},$$

i.e. the $i$th column of $X_\nu$ is the vector $\mathbf{x}_i^\nu$. The matrix $X_\nu$ is the transpose of the so-called shape-matrix discussed in detail in [11]. Note that the results presented in this letter do not depend on the particular ordering of the vertices, but it must be fixed for the $X_\nu$s to be well-defined.

### B. LP approach for CPA Lyapunov function

We now state the LP feasibility problem from [1] to parameterize a CPA CLF for the systems (1). Any solution to the problem can be used to parameterize a Lyapunov function for the switched system (1).

Assume $\mathcal{T}$ is a triangulation of a compact domain $\mathcal{D} \subset \mathbb{R}^n$ of the origin as described in the last section. We use two constants $\varepsilon_1, \varepsilon_2 > 0$ in the LP problem. Their values are of practical, but not theoretical interest, e.g. $\varepsilon_1 = \varepsilon_2 = 1$ is not restrictive for the theory; see [1] for details.

The variables of the LP problem are $V_\mathbf{x} \in \mathbb{R}$ for every vertex of a simplex in $\mathcal{T}$. The constraints of the LP problem are:

C1) $V_\mathbf{0} = 0$ and for every vertex $\mathbf{x}$ of a simplex in $\mathcal{T}$:

$$V_\mathbf{x} \ge \varepsilon_1\|\mathbf{x}\|_2 \tag{3}$$

C2) For every simplex $\mathfrak{S}_\nu := \mathrm{co}(\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu \dots, \mathbf{x}_n^\nu) \in \mathcal{T}$ define the vector of variables $\mathbf{v}_\nu = \left(V_{\mathbf{x}_1^\nu}, V_{\mathbf{x}_2^\nu}, \dots, V_{\mathbf{x}_n^\nu}\right)^T$ and recall that $X_\nu := (\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \dots, \mathbf{x}_n^\nu) \in \mathbb{R}^{n\times n}$.
The constraints are: for every simplex $\mathfrak{S}_\nu \in \mathcal{T}$, for all $j = 1, \dots, n$, and all $i = 1, 2, \dots, N$:

$$\mathbf{v}_\nu^{\mathrm{T}} X_\nu^{-1} A_i \mathbf{x}_j^\nu \le -\varepsilon_2\|\mathbf{x}_j^\nu\|_2. \tag{4}$$

Note that (4) is automatically fulfilled for $\mathbf{x}_0^\nu = \mathbf{0}$.

## C. Triangulations $\mathcal{T}^{std}$, $\mathcal{T}_K$, $K^{-1}\mathcal{T}_K$, and $\mathcal{T}_K^{\mathbf{F}}$

A suitable concrete triangulation for our aim of proving that the LP problem in Section II-B can always parameterize a CLF for the systems (1) when one exists, is the triangular-fan $\mathcal{T}_K$ of the triangulation in [8]; this is discussed in more detail in [1]. The triangulation is parameterized with $K \in \mathbb{N}_+$ and we show that if the origin is exponentially stable for the switched system (1), then for every $K \in \mathbb{N}_+$ large enough, the LP problem in Section II-B will have a feasible solution. In its definition we use the functions $\mathbf{R}^{\mathcal{J}} : \mathbb{R}^n \to \mathbb{R}^n$, defined for every $\mathcal{J} \subset \{1, 2, \dots, n\}$ by

$$\mathbf{R}^{\mathcal{J}}(\mathbf{x}) := \sum_{i=1}^{n} (-1)^{\chi_{\mathcal{J}}(i)} x_i \mathbf{e}_i, \quad \chi_{\mathcal{J}}(i) := \begin{cases} 1, & \text{if } i \in \mathcal{J}, \\ 0, & \text{if } i \notin \mathcal{J}. \end{cases}$$

where $\mathbf{e}_i$ is the standard $i$th unit vector in $\mathbb{R}^n$. Thus, $\mathbf{R}^{\mathcal{J}}(\mathbf{x})$ is the vector $\mathbf{x}$, except for a minus has been put in front of the coordinate $x_i$ whenever $i \in \mathcal{J}$.

We first define the triangulation $\mathcal{T}^{\text{std}}$ and then use it to construct the triangulations we will use for the LP problem.

The standard triangulation $\mathcal{T}^{\text{std}}$ consists of the simplices

$$\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \text{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \dots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right),$$

where

$$\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma} := \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{i=1}^{j} \mathbf{e}_{\sigma(i)}\right), \tag{5}$$

for all $\mathbf{z} \in \mathbb{N}_0^n$, all $\mathcal{J} \subset \{1, 2, \dots, n\}$, all $\sigma \in \text{Sym}(\{1, 2, \dots, n\})$, and $j = 0, 1, \dots, n$; note that $\mathbf{e}_{\sigma(i)}$, $j = \sigma(i)$, is the standard $j$th unit vector.

Now fix a $K \in \mathbb{N}_+$ and consider the simplices $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} \subset [-K, K]^n \subset \mathbb{R}^n$ in $\mathcal{T}^{\text{std}}$, that intersect the boundary of the hypercube $[-K, K]^n$. We are only interested in those intersections that are $(n-1)$-simplices. We take every simplex with vertices $\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma}$, $j \in \{0, 1, \dots, n\}$, where exactly one vertex $\mathbf{x}_{j^*}^{\mathbf{z}\mathcal{J}\sigma}$ satisfies $\|\mathbf{x}_{j^*}^{\mathbf{z}\mathcal{J}\sigma}\|_\infty < K$ and $\|\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma}\|_\infty = K$ for $j \in \{0, 1, \dots, n\} \setminus \{j^*\}$. Then we replace the vertex $\mathbf{x}_{j^*}^{\mathbf{z}\mathcal{J}\sigma}$ by $\mathbf{0}$; it is not difficult to see that $j^*$ is necessarily equal to 0. The set of the simplices constructed in this way triangulates $[-K, K]^n$ and this new triangulation, denoted $\mathcal{T}_K$, is our desired triangulation.

We will use two other triangulations, $K^{-1}\mathcal{T}_K$, and $\mathcal{T}_K^{\mathbf{F}}$, constructed from $\mathcal{T}_K$ by mapping the vertices of its simplices. Corresponding to the simplex $\mathfrak{S}_\nu = \text{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \dots, \mathbf{x}_n^\nu) \in \mathcal{T}_K$ is the simplex $\text{co}(K^{-1}\mathbf{x}_0^\nu, K^{-1}\mathbf{x}_1^\nu, \dots, K^{-1}\mathbf{x}_n^\nu) \in K^{-1}\mathcal{T}_K$ and the simplex $\text{co}(\mathbf{F}(\mathbf{x}_0^\nu), \mathbf{F}(\mathbf{x}_1^\nu), \dots, \mathbf{F}(\mathbf{x}_n^\nu)) \in \mathcal{T}_K^{\mathbf{F}}$, where $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and

$$\mathbf{F}(\mathbf{x}) := \frac{\|\mathbf{x}\|_\infty}{\|\mathbf{x}\|_2}\mathbf{x}, \quad \text{if } \mathbf{x} \neq \mathbf{0}. \tag{6}$$

The triangulations $K^{-1}\mathcal{T}_K$ and $\mathcal{T}_K^{\mathbf{F}}$ consist of exactly the simplices obtained in this manner from the simplices in $\mathcal{T}_K$. For a depiction of $\mathcal{T}_K^{\mathbf{F}}$ see [1]. The triangulation $\mathcal{T}_K^{\mathbf{F}}$ usually leads to LP problems with better numerical properties than when using $\mathcal{T}_K$ and the proof of Theorem 1 is more intuitive using the triangulation $K^{-1}\mathcal{T}_K$ rather than $\mathcal{T}_K$.

## III. Main results

Before we prove our main results in Theorem 1 and Theorem 2, we prove a few useful lemmas to shorten their proofs.

*Lemma 1:* Let $\mathbf{1} := (1, 1, \dots, 1)^{\mathrm{T}}, \mathbf{c} \in \mathbb{R}^n$, $c_1 \neq -1$, and $U = (u_{ij}) \in \mathbb{R}^{n \times n}$ with $u_{ij} = 1$ if $i \leq j$ and $u_{ij} = 0$ otherwise. Set $c_{n+1} := 0$ and define the vector $\mathbf{d} \in \mathbb{R}^n$ through

$$d_i = \frac{c_{i+1} - c_i}{c_1 + 1}, \quad \text{for} \quad i = 1, 2, \dots, n.$$

Then

$$\begin{aligned} &(U + \mathbf{c}\mathbf{1}^{\mathrm{T}})^{-1} = U^{-1} + \mathbf{d}\mathbf{e}_1^{\mathrm{T}}, \\ &\|(U + \mathbf{c}\mathbf{1}^{\mathrm{T}})^{-1}\|_1 \leq \max\{\|\mathbf{d}\|_1, 1\} + 1, \quad \text{and} \\ &\|(U + \mathbf{c}\mathbf{1}^{\mathrm{T}})^{-1}\|_\infty \leq \|\mathbf{d}\|_\infty + 2. \end{aligned}$$

*Proof:* Note that $U$ is invertible and the elements of $U^{-1} = (\widetilde{u}_{ij}) \in \mathbb{R}^{n \times n}$ are

$$\widetilde{u}_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -1, & \text{if } i + 1 = j, \\ 0, & \text{otherwise.} \end{cases}$$

For computing $(U + \mathbf{c}\mathbf{1}^{\mathrm{T}})^{-1}$ we use the Sherman-Morrison formula:

$$(U + \mathbf{c}\mathbf{1}^{\mathrm{T}})^{-1} = U^{-1} - \frac{U^{-1}\mathbf{c}\mathbf{1}^{\mathrm{T}}U^{-1}}{1 + \mathbf{1}^{\mathrm{T}}U^{-1}\mathbf{c}}.$$

Note that $U^{-1}\mathbf{c} = (c_1 - c_2, c_2 - c_3, \dots, c_n - c_{n+1})^{\mathrm{T}}$ and $\mathbf{1}^{\mathrm{T}}U^{-1} = \mathbf{e}_1^{\mathrm{T}}$, thus $1 + \mathbf{1}^{\mathrm{T}}U^{-1}\mathbf{c} = 1 + c_1 \neq 0$ and

$$\begin{aligned} \frac{U^{-1}\mathbf{c}\mathbf{1}^{\mathrm{T}}U^{-1}}{1 + \mathbf{1}^{\mathrm{T}}U^{-1}\mathbf{c}} &= \frac{(c_1 - c_2, c_2 - c_3, \dots, c_n - c_{n+1})^{\mathrm{T}}\mathbf{e}_1^{\mathrm{T}}}{1 + c_1} \\ &= -\mathbf{d}\mathbf{e}_1^{\mathrm{T}}. \end{aligned}$$

Thus

$$(U + \mathbf{c}\mathbf{1}^{\mathrm{T}})^{-1} = U^{-1} + \mathbf{d}\mathbf{e}_1^{\mathrm{T}}$$

by the Sherman-Morrison formula. The estimates for the norms follow immediately from the formula for the inverse. ∎

We now use the Lemma 1 to obtain upper bounds on norms of the inverses of the matrices $X_\nu$ for the simplices in $\mathcal{T}^{\text{std}}$.

*Corollary 1:* Let $\mathfrak{S}_\nu = \text{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \dots, \mathbf{x}_n^\nu) \in \mathcal{T}_K$ and set $X_\nu = (\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \dots, \mathbf{x}_n^\nu) \in \mathbb{R}^{n \times n}$. Then

$$\|X_\nu^{-1}\|_1 \leq 3, \quad \|X_\nu^{-1}\|_\infty \leq n + 1, \quad \|X_\nu^{-1}\|_2 \leq \sqrt{3(n+1)}.$$

*Proof:* By Section II-C, in particular formula (5), we may assume that there is a subset $\mathcal{J} \subset \{1, 2, \dots, n\}$, a permutation $\sigma \in \text{Sym}(\{1, 2, \dots, n\})$, and a vector $\mathbf{z} \in \mathbb{N}_0^n$ with $\|\mathbf{z}\|_\infty = K - 1$, such that for $k = 1, 2, \dots, n$ (recall that $\mathbf{x}_0^\nu = \mathbf{0}$) we have

$$\mathbf{x}_k^\nu = \mathbf{x}_k^{\mathbf{z}\mathcal{J}\sigma} = \mathbf{R}^{\mathcal{J}}\left(\mathbf{z} + \sum_{\ell=1}^{k} \mathbf{e}_{\sigma(\ell)}\right) = R_{\mathbf{J}}\mathbf{z} + R_{\mathbf{J}}\,\mathbf{u}_k^\sigma, \tag{7}$$

where $\mathbf{u}_k^\sigma := \sum_{\ell=1}^{k} \mathbf{e}_{\sigma(\ell)}$ and $R_{\mathbf{J}} \in \mathbb{R}^{n \times n}$ is the matrix representation of $\mathbf{R}^{\mathcal{J}}$, i.e. $R_{\mathbf{J}} := \text{diag}(J_1, J_2, \dots, J_n)$, with $J_i = -1$ if $i \in \mathcal{J}$ and $J_i = 1$ if $i \notin \mathcal{J}$.

Define the permutation matrix $P_\sigma \in \mathbb{R}^{n \times n}$, $P_\sigma = \left(\mathbf{e}_{\sigma(1)}, \mathbf{e}_{\sigma(2)}, \dots, \mathbf{e}_{\sigma(n)}\right)^{\mathrm{T}}$, i.e. $P_\sigma \mathbf{e}_{\sigma(i)} = \mathbf{e}_i$ and $P_\sigma^{\mathrm{T}}\mathbf{e}_i = \mathbf{e}_{\sigma(i)}$. Then $P_\sigma P_\sigma^{\mathrm{T}} = I$ (the identity matrix) and $\|P_\sigma\|_p = 1$

for $p \in \{1,2,\infty\}$. Hence, with $\mathbf{c} := P_\sigma \mathbf{z}$ we have $c_1 = K-1$ and

$$0 \le c_k \le K-1 \quad \text{and} \quad \mathbf{y}_k := P_\sigma R_{\mathbf{J}} \mathbf{x}_k^\nu = \mathbf{c} + \sum_{\ell=1}^{k} \mathbf{e}_\ell$$

for $k = 1,2,\ldots,n$. That is, $P_\sigma R_{\mathbf{J}} X_\nu = U + \mathbf{c}\mathbf{1}^{\mathrm{T}}$, where the matrix on the right-hand-side is of the same form as in Lemma 1. Further, the vector $\mathbf{d} = (d_1, d_2, \ldots, d_n)^{\mathrm{T}}$ in Lemma 1 fulfills $|d_k| \le (K-1)/K \le 1$ for $k = 1,2,\ldots,n$. Therefore, because $P_\sigma^{\mathrm{T}} = P_\sigma^{-1}$ and $R_{\mathbf{J}} = R_{\mathbf{J}}^{-1}$, and for $p \in \{1,2,\infty\}$, we have that

$$\begin{aligned}\|X_\nu^{-1}\|_p &= \|X_\nu^{-1} R_{\mathbf{J}}^{-1} P_\sigma^{-1} P_\sigma R_{\mathbf{J}}\|_p \\ &\le \|(P_\sigma R_{\mathbf{J}} X_\nu)^{-1}\|_p \|P_\sigma\|_p \|R_{\mathbf{J}}\|_p \\ &= \|(U + \mathbf{c}\mathbf{1}^{\mathrm{T}})^{-1}\|_p \cdot 1 \cdot 1.\end{aligned}$$

Since $\|\mathbf{d}\|_1 \le n$ and $\|\mathbf{d}\|_\infty \le 1$ it follows from Lemma 1 that

$$\|X_\nu^{-1}\|_1 \le n+1 \quad \text{and} \quad \|X_\nu^{-1}\|_\infty \le 3$$

and the bound

$$\|X_\nu^{-1}\|_2 \le \sqrt{3(n+1)}$$

follows from the well known $\|A\|_2^2 \le \|A\|_1 \|A\|_\infty$ for any $A \in \mathbb{R}^{n\times n}$. ■

The next lemma adapts results from [16] to deliver a CLF for the systems (1), that has all the properties needed to be approximated arbitrary close in the $C^1$ norm by CPA functions. This is already done in [16], but not for CPA functions with a priori fixed triangulations as we need to prove that the LP approach to compute a CLF always succeeds when a CLF exists for the systems (1). Recall from the Introduction that the existence of a CLF for the systems (1) is equivalent to the exponential stability of the origin for the arbitrary switched system $\dot{\mathbf{x}} \in \mathrm{co}\{A_i\mathbf{x}\}$.

*Lemma 2:* Assume the origin is exponentially stable for the arbitrary switched system $\dot{\mathbf{x}} \in \mathrm{co}\{A_i\mathbf{x}\}$ corresponding to the systems (1). Then there exists a continuous $V\colon \mathbb{R}^n \to \mathbb{R}$ fulfilling:

1) $V \in C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\})$.
2) There exist constants $\alpha_1, \alpha_2, \alpha_3 > 0$ such that
$$\alpha_1 \|\mathbf{x}\|_2 \le V(\mathbf{x}) \le \alpha_2 \|\mathbf{x}\|_2 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \tag{8}$$
and for all $i = 1,2,\ldots,N$ we have
$$\nabla V(\mathbf{x}) A_i \mathbf{x} \le -\alpha_3 \|\mathbf{x}\|_2 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \tag{9}$$
3) For every $c > 0$ we have
$$V(c\mathbf{x}) = cV(\mathbf{x}) \quad \text{and} \quad [\nabla V](c\mathbf{x}) = \nabla V(\mathbf{x})$$
for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.
4)
$$V(\mathbf{x}) = \nabla V(\mathbf{x})\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

In particular, $V$ is a CLF for the systems (1).

*Proof:* By [16] there exist a compact, strictly convex neighbourhood $\emptyset \neq \mathcal{W} \subset \mathbb{R}^n$ of the origin, constants $p, M \in \mathbb{N}_+$, $\alpha > 0$, and vectors $\mathbf{g}_j \in \mathbb{R}^n$, $j = 1,2,\ldots,M$, such that

$$W(\mathbf{x}) = \sum_{j=1}^{M} \left(\mathbf{g}_j^{\mathrm{T}}\mathbf{x}\right)^{2p} \text{ fulfills } \max_{\substack{\mathbf{x}\in\partial\mathcal{W} \\ i=1,2,\ldots,N}} \nabla W(\mathbf{x}) A_i \mathbf{x} \le -\alpha.$$

Further, $W(\mathbf{x}) > 0$ if $\mathbf{x} \neq \mathbf{0}$. Note that in [16] our $W$ is denoted $\widetilde{W}$ and our $\mathcal{W}$ is a sublevel set $\{\mathbf{x} \in \mathbb{R}^n \colon W(\mathbf{x}) \le 1\}$ of a continuous, positively homogenous CLF $W\colon \mathbb{R}^n \to \mathbb{R}$ (not our $W$).

First, we prove a few properties for $W$. From the formulas

$$W(\mathbf{x}) = \sum_{j=1}^{M} \left(\mathbf{g}_j^{\mathrm{T}}\mathbf{x}\right)^{2p} \text{ and } \nabla W(\mathbf{x}) = 2p \sum_{j=1}^{N} \left(\mathbf{g}_j^{\mathrm{T}}\mathbf{x}\right)^{2p-1} \mathbf{g}_j^{\mathrm{T}}$$

we clearly have

$$W(c\mathbf{x}) = c^{2p} W(\mathbf{x}) \quad \text{and} \quad [\nabla W](c\mathbf{x}) = c^{2p-1} \nabla W(\mathbf{x}) \tag{10}$$

for $c > 0$ and $\mathbf{x} \neq \mathbf{0}$.

Let $r, R > 0$ be such that $r \le \|\widetilde{\mathbf{x}}\|_2 \le R$ for all $\widetilde{\mathbf{x}} \in \partial\mathcal{W}$ and define

$$0 < m := \min_{\mathbf{x}\in\partial\mathcal{W}} W(\mathbf{x}) \le \max_{\mathbf{x}\in\partial\mathcal{W}} W(\mathbf{x}) =: M.$$

Fix an arbitrary $\mathbf{x} \neq \mathbf{0}$. Then there is a constant $c_x > 0$ and a unique $\widetilde{\mathbf{x}} \in \partial\mathcal{W}$ such that $\mathbf{x} = c_x \widetilde{\mathbf{x}}$ and

$$c_x r \le \|\mathbf{x}\|_2 = c_x \|\widetilde{\mathbf{x}}\|_2 \le c_x R, \quad \text{i.e. } \frac{\|\mathbf{x}\|_2}{R} \le c_x \le \frac{\|\mathbf{x}\|_2}{r}.$$

Combined with (10) this gives

$$\begin{aligned} m\frac{\|\mathbf{x}\|_2^{2p}}{R^{2p}} &\le m c_x^{2p} \le W(\mathbf{x}) = W(c_x\widetilde{\mathbf{x}}) \\ &= c_x^{2p} W(\widetilde{\mathbf{x}}) \le M c_x^{2p} \le M \frac{\|\mathbf{x}\|_2^{2p}}{r^{2p}} \end{aligned} \tag{11}$$

and, for every $i = 1,2,\ldots,N$,

$$\begin{aligned} \nabla W(\mathbf{x}) A_i \mathbf{x} &= [\nabla W](c_x\widetilde{\mathbf{x}}) A_i (c_x\widetilde{\mathbf{x}}) \\ &= c_x^{2p} \nabla W(\widetilde{\mathbf{x}}) A_i \widetilde{\mathbf{x}} \le -\alpha c_x^{2p} \le -\frac{\alpha}{R^{2p}} \|\mathbf{x}\|_2^{2p}. \end{aligned} \tag{12}$$

We define $V\colon \mathbb{R}^n \to \mathbb{R}$ through $V(\mathbf{x}) := [W(\mathbf{x})]^{\frac{1}{2p}}$ and show that it has all the claimed properties.

**1)** Clearly $V$ is continuous and $V \in C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\})$.

**2)** From (11) it follows that

$$\frac{m^{\frac{1}{2p}}}{R} \|\mathbf{x}\|_2 \le V(\mathbf{x}) \le \frac{M^{\frac{1}{2p}}}{r} \|\mathbf{x}\|_2$$

for all $\mathbf{x} \in \mathbb{R}^n$, i.e. (8) with $\alpha_1 := m^{\frac{1}{2p}}/R$ and $\alpha_2 := M^{\frac{1}{2p}}/r$. From (12) it follows that for every $i = 1,2,\ldots,N$ and every $\mathbf{x} \neq \mathbf{0}$ we have

$$\begin{aligned} \nabla V(\mathbf{x}) A_i \mathbf{x} &= \frac{1}{2p} [W(\mathbf{x})]^{\frac{1}{2p}-1} \nabla W(\mathbf{x}) A_i \mathbf{x} \\ &\le -\frac{1}{2p} [W(\mathbf{x})]^{\frac{1}{2p}-1} \frac{\alpha}{R^{2p}} \|\mathbf{x}\|_2^{2p} \\ &\le -\frac{1}{2p} \left(\frac{M}{r^{2p}} \|\mathbf{x}\|_2^{2p}\right)^{\frac{1}{2p}-1} \frac{\alpha}{R^{2p}} \|\mathbf{x}\|_2^{2p} \\ &\le -\frac{\alpha M^{\frac{1}{2p}-1}}{2pr} \left(\frac{r}{R}\right)^{2p} \|\mathbf{x}\|_2^{2p} \end{aligned}$$

i.e. (9) with $\alpha_3 := \alpha M^{\frac{1}{2p}-1} (r/R)^{2p} / (2pr)$.

**3 and 4)** By (10) we have for $\mathbf{x} \neq \mathbf{0}$ and $c > 0$ that

$$V(c\mathbf{x}) = [W(c\mathbf{x})]^{\frac{1}{2p}} = \left[c^{2p} W(\mathbf{x})\right]^{\frac{1}{2p}} = cV(\mathbf{x}).$$

Hence $V$ is positively homogeneous of order one. The two remaining statements of the lemma now follow immediately from Euler's Homogenous Function Theorem. ∎

We first prove our main results using the triangulation $K^{-1}\mathcal{T}_K$ in Theorem 1. Subsequently, we use Theorem 1 to prove the same for the triangulation $\mathcal{T}_K^{\mathbf{F}}$ in Theorem 2.

*Theorem 1:* Assume the origin is exponentially stable for the arbitrary switched system $\dot{\mathbf{x}} \in \text{co}\{A_i\mathbf{x}\}$ corresponding to the systems (1). Then there exists a $K^* \in \mathbb{N}_+$ such that the LP problem in Section II-B using the triangulation $\mathcal{T} = K^{-1}\mathcal{T}_K$ has a feasible solution whenever $K \geq K^*$.

*Proof:* Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be the Lyapunov function from Lemma 2 and define $c := \max\{\varepsilon_1/\alpha_1, 2\varepsilon_2/\alpha_3\}$. For every vertex $\mathbf{x}$ of a simplex in the triangulation $\mathcal{T} = K^{-1}\mathcal{T}_K$ we set the variable $V_\mathbf{x}$ from the LP problem in Section II-B equal to $cV(\mathbf{x})$ and show that the constraints C1 and C2 are fulfilled if $K \in \mathbb{N}_+$ is large enough.

For every such vertex $\mathbf{x}$ we have by (8)

$$V_\mathbf{x} = cV(\mathbf{x}) \geq \frac{\varepsilon_1}{\alpha_1}V(\mathbf{x}) \geq \frac{\varepsilon_1}{\alpha_1}\alpha_1\|\mathbf{x}\|_2 \geq \varepsilon_1\|\mathbf{x}\|_2 \tag{13}$$

so the constraints C1 are fulfilled.

Let $\mathfrak{S}_\nu := \text{co}(\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu \ldots, \mathbf{x}_n^\nu)$ be an arbitrary simplex in $\mathcal{T} = K^{-1}\mathcal{T}_K$ and set $X_\nu = (\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu) \in \mathbb{R}^{n\times n}$. Note that, e.g. by using formula (7) to represent $K\mathbf{x}_k^\nu$ and $K\mathbf{x}_\ell^\nu$, that

$$\|\mathbf{x}_k^\nu\|_\infty = 1 \;\text{ and }\; \|\mathbf{x}_k^\nu - \mathbf{x}_\ell^\nu\|_2 \leq \frac{\sqrt{n-1}}{K} \leq \frac{\sqrt{n}}{K} \tag{14}$$

for all $k, \ell = 1, 2, \ldots, n$. Further, we have

$$\|X_\nu^{-1}\|_2 \leq K\sqrt{3(n+1)}, \tag{15}$$

because $K\mathfrak{S}_\nu = \text{co}(\mathbf{0}, K\mathbf{x}_1^\nu, K\mathbf{x}_2^\nu \ldots, K\mathbf{x}_n^\nu) \in \mathcal{T}_K$ and thus $\|(K\mathbf{x}_1^\nu, K\mathbf{x}_2^\nu \ldots, K\mathbf{x}_n^\nu)^{-1}\|_2 \leq \sqrt{3(n+1)}$ by Corollary 1. With the vector of variables $\mathbf{v}_\nu = \left(V_{\mathbf{x}_1^\nu}, V_{\mathbf{x}_2^\nu}, \ldots, V_{\mathbf{x}_n^\nu}\right)^{\mathrm{T}}$ we have

$$\mathbf{v}_\nu := c\cdot\left(V(\mathbf{x}_1^\nu), V(\mathbf{x}_2^\nu), \ldots, V(\mathbf{x}_n^\nu)\right)^{\mathrm{T}}$$

with our assigned values to the variables.

For the rest of the proof let $j \in \{1, 2, \ldots, n\}$ be arbitrary, but fixed. We have by (15),

$$\begin{aligned}\|\mathbf{v}_\nu^{\mathrm{T}}X_\nu^{-1} - c\nabla V(\mathbf{x}_j^\nu)\|_2 &= \|[\mathbf{v}_\nu^{\mathrm{T}} - c\nabla V(\mathbf{x}_j^\nu)X_\nu]X_\nu^{-1}\|_2 \\ &\leq \|\mathbf{v}_\nu^{\mathrm{T}} - c\nabla V(\mathbf{x}_j^\nu)X_\nu\|_2\|X_\nu^{-1}\|_2 \qquad (16) \\ &\leq \|\mathbf{v}_\nu^{\mathrm{T}} - c\nabla V(\mathbf{x}_j^\nu)X_\nu\|_2 \cdot K\sqrt{3(n+1)}.\end{aligned}$$

Denote by $H\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ the Hessian matrix of the function $V$. By Taylor's theorem and because $V(\mathbf{x}_j^\nu) = \nabla V(\mathbf{x}_j^\nu)\mathbf{x}_j^\nu$ we have for every $k = 1, 2, \ldots, n$, $k \neq j$, that

$$\begin{aligned}V(\mathbf{x}_k^\nu) &= V(\mathbf{x}_j^\nu) + \nabla V(\mathbf{x}_j^\nu)(\mathbf{x}_k^\nu - \mathbf{x}_j^\nu) \qquad (17) \\ &\quad + \frac{1}{2}(\mathbf{x}_k^\nu - \mathbf{x}_j^\nu)^{\mathrm{T}}H(\boldsymbol{\xi})(\mathbf{x}_k^\nu - \mathbf{x}_j^\nu) \\ &= \nabla V(\mathbf{x}_j^\nu)\mathbf{x}_k^\nu + \frac{1}{2}(\mathbf{x}_k^\nu - \mathbf{x}_j^\nu)^{\mathrm{T}}H(\boldsymbol{\xi})(\mathbf{x}_k^\nu - \mathbf{x}_j^\nu)\end{aligned}$$

for some $\boldsymbol{\xi}$ on the line segment between $\mathbf{x}_j^\nu$ and $\mathbf{x}_k^\nu$. Set $\widetilde{H} := \frac{1}{2}\max_{\|\mathbf{x}\|_\infty=1}\|H(\mathbf{x})\|_2$. The $k$th component of the vector $c^{-1}\mathbf{v}^{\mathrm{T}} - \nabla V(\mathbf{x}_j^\nu)X_\nu$ is $V(\mathbf{x}_k^\nu) - \nabla V(\mathbf{x}_j^\nu)\mathbf{x}_k^\nu$ and can be bounded using (14) and (17) by

$$|V(\mathbf{x}_k^\nu) - \nabla V(\mathbf{x}_j^\nu)\mathbf{x}_k^\nu| \leq \widetilde{H}\|\mathbf{x}_k^\nu - \mathbf{x}_j^\nu\|_2^2 \leq \frac{n\widetilde{H}}{K^2}.$$

Combined with (16) this delivers

$$\begin{aligned}\|\mathbf{v}_\nu^{\mathrm{T}}X_\nu^{-1} - c\nabla V(\mathbf{x}_j^\nu)\|_2 &\leq \sqrt{n}\cdot\frac{n\widetilde{H}}{K^2}\cdot cK\sqrt{3(n+1)} \\ &= n\sqrt{3n(n+1)}\widetilde{H}cK^{-1}.\end{aligned}$$

It follows, by using (9) and since $c \geq 2\varepsilon_2/\alpha_3$, that

$$\begin{aligned}\mathbf{v}_\nu^{\mathrm{T}}X_\nu^{-1}A_i\mathbf{x}_j^\nu &= c\nabla V(\mathbf{x}_j^\nu)A_i\mathbf{x}_j^\nu \\ &\quad + \left[\mathbf{v}_\nu^{\mathrm{T}}X_\nu^{-1} - c\nabla V(\mathbf{x}_j^\nu)\right]A_i\mathbf{x}_j^\nu \\ &\leq -c\alpha_3\|\mathbf{x}_j^\nu\|_2 \\ &\quad + \|\mathbf{v}_\nu^{\mathrm{T}}X_\nu^{-1} - c\nabla V(\mathbf{x}_j^\nu)\|_2\|A_i\mathbf{x}_j^\nu\|_2 \\ &\leq -2\varepsilon_2\|\mathbf{x}_j^\nu\|_2 \\ &\quad + n\sqrt{3n(n+1)}\widetilde{H}cK^{-1}\|A_i\|_2\|\mathbf{x}_j^\nu\|_2.\end{aligned}$$

Thus, choosing $K^*$ so large that

$$K^* \geq \frac{cn\sqrt{3n(n+1)}\widetilde{H}}{\varepsilon_2}\cdot\max_{i=1,2,\ldots,N}\|A_i\|_2$$

ensures constraints C2 are fulfilled for every $K \geq K^*$. ∎

We now use Theorem 1 to prove that we can just as well use the numerically more adequate triangulation $\mathcal{T}_K^{\mathbf{F}}$ in the LP problem in Section II-B and we still are guarantied to get a solution.

*Theorem 2:* Assume the origin is exponentially stable for the arbitrary switched system $\dot{\mathbf{x}} \in \text{co}\{A_i\mathbf{x}\}$ corresponding to the systems (1). Then there exists a $K^* \in \mathbb{N}_+$ such that the LP problem in Section II-B using the triangulation $\mathcal{T} = \mathcal{T}_K^{\mathbf{F}}$ has a feasible solution whenever $K \geq K^*$.

*Proof:* Let $K^*$ be as in Theorem 1 and let $K \geq K^*$. Let $\mathfrak{S}_\nu = \text{co}(\mathbf{0}, \mathbf{y}_1^\nu, \mathbf{y}_2^\nu \ldots, \mathbf{y}_n^\nu) \in \mathcal{T}_K^{\mathbf{F}}$ be arbitrary. Then there is a simplex $\text{co}(\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu \ldots, \mathbf{x}_n^\nu) \in K^{-1}\mathcal{T}_K$ such that

$$\mathbf{y}_j^\nu := \frac{\|K\mathbf{x}_j^\nu\|_\infty}{\|K\mathbf{x}_j^\nu\|_2}K\mathbf{x}_j^\nu = F_j^\nu\mathbf{x}_j^\nu, \text{ with } F_j^\nu := \frac{K\|\mathbf{x}_j^\nu\|_\infty}{\|\mathbf{x}_j^\nu\|_2}\mathbf{x}_j^\nu$$

for $j = 1, 2, \ldots, n$. With

$$\begin{aligned}Y_\nu &:= (\mathbf{y}_1^\nu, \mathbf{y}_2^\nu, \ldots, \mathbf{y}_n^\nu) \in \mathbb{R}^{n\times n}, \\ X_\nu &:= (\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu) \in \mathbb{R}^{n\times n}, \text{ and} \\ F_\nu &:= \text{diag}(F_1^\nu, F_2^\nu, \ldots, F_n^\nu) \in \mathbb{R}^{n\times n},\end{aligned}$$

we have $Y_\nu = X_\nu F_\nu$ and $F_\nu^{\mathrm{T}} = F_\nu$. In particular, with $V_\mathbf{x} := cV(\mathbf{x})$, as in the proof of Theorem 1, and

$$\mathbf{v}_\nu := c\cdot\left(V(\mathbf{x}_1^\nu), V(\mathbf{x}_2^\nu), \ldots, V(\mathbf{x}_n^\nu)\right)^{\mathrm{T}},$$

we have

$$\mathbf{v}_\nu^\mathbf{y} := c\cdot\left(V(\mathbf{y}_1^\nu), V(\mathbf{y}_2^\nu), \ldots, V(\mathbf{y}_n^\nu)\right)^{\mathrm{T}} = F_\nu\mathbf{v}_\nu$$

because $V(\mathbf{y}_j^\nu) = V(F_j^\nu\mathbf{x}_j^\nu) = F_j^\nu V(\mathbf{x}_j^\nu)$ by Lemma 2.

We conclude the proof by showing that the constraints of the LP problem in Section II-B are fulfilled for the $\mathbf{y}_j^\nu$ because they are fulfilled for the $\mathbf{x}_j^\nu$, $j = 1, 2, \ldots, n$.

The constraints C1 follow from

$$V_{\mathbf{y}_j^\nu} = cV(F_j^\nu \mathbf{x}_j^\nu) = F_j^\nu cV(\mathbf{x}_j^\nu) = F_j^\nu V_{\mathbf{x}_j^\nu}$$
$$\geq F_j^\nu \varepsilon_1 \|\mathbf{x}_j^\nu\|_2 = \varepsilon_1 \|F_j^\nu \mathbf{x}_j^\nu\|_2 = \varepsilon_1 \|\mathbf{y}_j^\nu\|_2.$$

The constraints C2 follow from

$$(\mathbf{v}_\nu^{\mathbf{y}})^{\mathrm{T}} Y_\nu^{-1} A_i \mathbf{y}_j^\nu = \mathbf{v}_\nu^{\mathrm{T}} F_\nu^{\mathrm{T}} F_\nu^{-1} X_\nu^{-1} A_i F_j^\nu \mathbf{x}_j^\nu$$
$$= F_j^\nu \mathbf{v}_\nu^{\mathrm{T}} X_\nu^{-1} A_i \mathbf{x}_j^\nu$$
$$\leq -F_j^\nu \varepsilon_2 \|\mathbf{x}_j^\nu\|_2 = -\varepsilon_2 \|\mathbf{y}_j^\nu\|_2.$$

■

## IV. Conclusions

We proved, that the linear programming (LP) based approach from [1] for linear systems $\dot{\mathbf{x}} = A_i\mathbf{x}$, $i = 1, 2, \ldots, N$, always succeeds in computing a common Lyapunov function (CLF) if the corresponding arbitrary switched system $\dot{\mathbf{x}} \in \mathrm{co}\{A_i\mathbf{x}\}$ has an exponentially stable equilibrium at the origin. In more detail, we proved in Theorem 2 that if the triangulation used by the method is fine enough, specified by a parameter $K \in \mathbb{N}_+$, then the resulting LP problem has a feasible solution. Thus, the method in [1], which computes continuous and piecewise affine (CPA) CLFs is not only more general than searching for a quadratic common Lyapunov functions (QCLFs) using linear matrix inequalities (LMIs), as suggest by the examples in [1], but is not limiting at all.

Although these theoretical results are very satisfactory, the curse of dimensionality remains a limiting factor in practice. It remains to be investigated how this practical problem can be eased, e.g. by using a kind of preconditioning as in [1], [2] and/or by using specific kinds of CPA functions parameterized with fewer parameters. This will be the subject of further research.

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