Contraction Metric Computation using numerical Integration and Quadrature

Peter Giesl*[1], Sigurdur Hafstein†[2], and Iman Mehrabinezhad‡[3]

[1]Department of Mathematics, University of Sussex, Falmer BN1 9QH, United Kingdom
[2]Faculty of Physical Sciences, University of Iceland, Dunhagi 5, IS-107 Reykjavik, Iceland
[3]CARE-C Center, The Cyprus Institute, 20 Konstantinou Kavafi St., 2121, Nicosia, Cyprus

September 25, 2023

Abstract

We present a novel method to compute contraction metrics for general nonlinear dynamical systems with exponentially stable equilibria. Such a contraction metric delivers information on the long term behaviour of the system and is robust with respect to perturbations of the dynamics, even perturbations that shift the equilibrium. We prove that our method is always able to deliver a contraction metric in any compact subset of the basin of attraction of an exponentially stable equilibrium. Further, we demonstrate the applicability of the method by computing contraction metrics for two three-dimensional systems from the literature.

Keywords: differential equation, contraction metric, equilibrium, exponential stability, rigorous numerical method, continuous piecewise affine functions

Mathematics Subject Classification **(MSC2020): 65L07, 37B25, 65L06, 34D20**

1 Introduction

Consider the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{f} \in C^s(\mathbb{R}^n; \mathbb{R}^n), \quad s \geq 1. \tag{1.1}$$

The solution $\mathbf{x}(t)$ to the initial value problem (1.1) with $\mathbf{x}(0) = \boldsymbol{\xi}$ is denoted by $\boldsymbol{\phi}(t, \boldsymbol{\xi})$.

For studying the long term behaviour of the system (1.1) one usually tries to identify its (local) attractors $\mathcal{A}_i \subset \mathbb{R}^n$ and their corresponding basins of attraction $\mathcal{B}_i := \{\mathbf{x} \in \mathbb{R}^n : \lim_{t\to\infty} d(\boldsymbol{\phi}(t, \mathbf{x}), \mathcal{A}_i) = 0\}$; here $d(\mathbf{x}, \mathcal{A}_i)$ denotes the distance between the point $\mathbf{x} \in \mathbb{R}^n$ and the

*email: p.a.giesl@sussex.ac.uk
†email: shafstein@hi.is
‡email: i.mehrabinzhad@cyi.ac.cy

set $\mathcal{A}_i$. Other important concepts for the long term behaviour are e.g. positively invariant sets $\mathcal{F}$, fulfilling $\phi([0,\infty),\mathcal{F}) \subset \mathcal{F}$, repellers, i.e. attractors for the time-reversed flow, the chain-recurrent set, which contains all equilibria and periodic orbits, or the types of attractors, e.g. stable equilibria, stable periodic orbits, or strange attractors; see e.g. [37, 8, 53].

For the long term analysis, it has proved itself very useful to study either real-valued functions from the phase-space of the system that are decreasing along solution trajectories, so-called (complete) Lyapunov functions [48, 56, 5, 13, 40, 41, 2, 7, 35], or Finsler structures/Riemannian metrics adapted to the dynamics, so-called Finsler-Lyapunov functions or contraction metrics [46, 45, 14, 44, 11, 38, 39, 47, 4, 54, 17, 19]. While Lyapunov functions are scalar-valued functions, Riemannian contraction metrics are matrix-valued functions. However, to find a Lyapunov function for an equilibrium, the position of the equilibrium is needed a priori, while the conditions for a contraction metric are independent of the equilibrium and, moreover, even robust with respect to perturbations of the location of the equilibrium.

Since the analytic computation of Lyapunov functions or contraction metrics is very difficult or impossible, except in the simplest cases, numerous numerical methods for their computation have emerged; for an overview of these methods see e.g. the reviews [24] for Lyapunov functions and [26] for contraction metrics. The methods for Lyapunov functions range from numerically approximating solutions to the Zubov PDE [57] by parameterizing rational functions [55] or using collocation with radial basis functions in reproducing kernel Hilbert spaces [18], over parameterizations of Lyapunov functions using linear programming [43, 42, 49, 22], to computing polynomial Lyapunov functions using semidefinite optimization [51, 12, 52]. Some of these methods have also been adapted and extended for the computation of contraction metrics [6, 21, 20].

In this paper we develop a novel method for the computation of contraction metrics for the system (1.1). We first use numerical solutions to initial value problems as well as numerical quadrature to compute the contraction metric at points of a simplicial complex. Then, in the verification step, we interpolate these values to obtain a CPA (continuous piecewise affine) metric and then rigorously verify that this metric is indeed contracting.

Previously, methods have been proposed to compute a CPA contraction metric directly, using semidefinite programming in the case of time-periodic systems [21], or, using meshfree collocation in the first step and then CPA verification [27, 28]. The numerical solutions in the first step of our novel method can be regarded as a nontrivial extension of the Lyapunov function computation methods in [9, 34, 10, 16, 15] to the computation of contraction metrics. To state our method concretely, we will use the Adams-Bashforth method of order 4 (AB4), initialized with the usual Runge-Kutta method of order 4 (RK4), for the numerical integration of initial-value problems and the Composite Trapezoidal Rule and a Romberg-like extrapolation for the quadrature. It is, however, not difficult to see that our method works with every numerical integrator with local truncation error of order 2 or higher.

Notation: We denote the usual p-norms on $\mathbb{R}^n$ and the corresponding induced matrix norms by $\|\cdot\|_p$, $1 \le p < \infty$. For both vectors in $\mathbb{R}^n$ and matrices in $\mathbb{R}^{n\times n}$ we write $\|\cdot\|_{\max}$ for the maximum absolute value norm, i.e. $\|\mathbf{x}\|_{\max} := \max_{i=1,2,\dots,n} |x_i|$ for a vector $\mathbf{x} \in \mathbb{R}^{n\times n}$ and $\|A\|_{\max} := \max_{i,j=1,2,\dots,n} |a_{ij}|$ for a matrix $A = \big(a_{ij}\big)_{i,j=1,\dots,n} \in \mathbb{R}^{n\times n}$. Apart from the usual equivalence estimates for the p-norms on $\mathbb{R}^n$, recall the norm equivalence $\|A\|_{\max} \le \|A\|_2 \le n\|A\|_{\max}$ for a matrix $A \in \mathbb{R}^{n\times n}$. We denote the symmetric $n \times n$ matrices with real entries by $\mathbb{S}^{n\times n}$. We use the C^k-norm for $W \in C^k(\mathcal{D};\mathcal{R})$, defined as

$$\|W\|_{C^k(\mathcal{D};\mathcal{R})} := \sum_{|\boldsymbol{\alpha}|\le k} \sup_{\mathbf{x}\in\mathcal{D}} \|D^{\boldsymbol{\alpha}} W(\mathbf{x})\|_2 .$$

Here $\mathcal{D} \subset \mathbb{R}^n$ is a non-empty open set, $\mathcal{R}$ is one of $\mathbb{R}, \mathbb{R}^n, \mathbb{S}^{n\times n}$, or $\mathbb{R}^{n\times n}$, and $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ is a multi-index with length $|\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha_i$. We denote by I the identity matrix and by $D\mathbf{f}(\mathbf{x}) \in \mathbb{R}^{n\times n}$ the Jacobian matrix of $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ at $\mathbf{x} \in \mathbb{R}^n$. Finally, we define $\mathbb{N}_+ := \{1, 2, 3, \ldots\}$ and $\mathbb{R}_+ := [0, \infty)$.

Let us give an overview of the paper: In Section 1.1 we recall some facts about contraction metrics, including their use to determine subsets of the basin of attraction of an equilibrium and an existence result. In Section 1.2 we recall in Theorem 1.8 a uniform error estimate for an approximation to a contraction metric computed using numerical integration and quadrature. In Section 2 we deal with the second step of the method, namely the CPA interpolation of the values computed in the first step, and prove in Theorem 2.11 that the combined method is always successful in computing and verifying a contraction metric when the numerical approximation is sufficiently accurate (long enough time interval and small enough time steps) and the points used for the interpolation are sufficiently dense. Section 3 applies the method to two 3-dimensional examples, before we conclude in Section 4.

1.1 Contraction Metrics

In this section we will review basic concepts about Riemannian contraction metrics and some important tools that we will use in this paper. Since the Riemannian metric we calculate later will not be differentiable, we give a suitable non-smooth definition.

1.1 Definition (Riemannian metric)
Let G be an open subset of $\mathbb{R}^n$. A Riemannian metric is a locally Lipschitz continuous matrix-valued function $M : G \to \mathbb{S}^{n\times n}$, such that $M(\mathbf{x})$ is positive definite for all $\mathbf{x} \in G$.

The forward orbital derivative $M'_+(\mathbf{x})$ with respect to (1.1) at $\mathbf{x} \in G$ is defined by

$$M'_+(\mathbf{x}) := \limsup_{h\to 0^+} \frac{M(\boldsymbol{\phi}(h,\mathbf{x})) - M(\mathbf{x})}{h}. \tag{1.2}$$

With a Riemannian metric M we can define a (point-dependent) scalar product for each $\mathbf{x} \in G$ through $\langle \mathbf{v}, \mathbf{w}\rangle_{M(\mathbf{x})} := \mathbf{v}^T M(\mathbf{x})\mathbf{w}$ for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

1.2 Remark Note that the forward orbital derivative (1.2) is formulated using a Dini derivative similar to [21, Definition 3.1] and always exists in $\mathbb{R} \cup \{\infty\}$. This assumption is less restrictive than [19, Definition 2.1], which is the existence and continuity of

$$M'(\mathbf{x}) = \left.\frac{d}{dt} M(\boldsymbol{\phi}(t,\mathbf{x}))\right|_{t=0}.$$

A sufficient condition for the existence and continuity of $M'(\mathbf{x})$ is that $M \in C^1(G;\mathbb{S}^{n\times n})$; then $M'_+(\mathbf{x}) = M'(\mathbf{x})$, where $(M'(\mathbf{x}))_{ij} = \nabla M_{ij}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$ for all $i, j \in \{1, 2, \ldots, n\}$. It is also worth mentioning that if $K \subset G$ is compact, then M in Definition 1.1 is uniformly positive definite on K, i.e. there exists an $\varepsilon > 0$ such that $\mathbf{v}^T M(\mathbf{x})\mathbf{v} \ge \varepsilon\|\mathbf{v}\|_2^2$ for all $\mathbf{v} \in \mathbb{R}^n$ and all $\mathbf{x} \in K$.

1.3 Remark It is useful to have a more accessible expression for the forward orbital derivative in terms of $\mathbf{f}$ in (1.1). In fact we have

$$M'_+(\mathbf{x}) := \limsup_{h\to 0^+} \frac{M\big(\boldsymbol{\phi}(h,\mathbf{x})\big) - M\big(\mathbf{x}\big)}{h} = \limsup_{h\to 0^+} \frac{M\big(\mathbf{x}+h\mathbf{f}(\mathbf{x})\big) - M(\mathbf{x})}{h},$$

because by [21, Lemma 3.3] an analogous formula holds true for each (locally Lipschitz) entry M_{ij} of the matrix M.

1.4 Definition (Riemannian contraction metric) *[19, Definition 2.4]*
Let K be a compact subset of an open set $G \subset \mathbb{R}^n$ and $M \in C^0(G; \mathbb{S}^{n\times n})$ be a Riemannian metric. For $\mathbf{x} \in G$ and $\mathbf{v} \in \mathbb{R}^n$ define

$$L_M(\mathbf{x};\mathbf{v}) := \frac{1}{2}\mathbf{v}^T\big[M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'_+(\mathbf{x})\big]\mathbf{v}$$

with respect to the system (1.1)*.*

The Riemannian metric M is called contracting in $K \subset G$ with exponent $-\nu < 0$, or a contraction metric on K, if

$$\mathcal{L}_M(\mathbf{x}) \;\; := \max_{\mathbf{v}^T M(\mathbf{x})\mathbf{v}=1} L_M(\mathbf{x};\mathbf{v}) \leq -\nu \text{ for all } \mathbf{x}\in K. \tag{1.3}$$

1.5 Remark Fix $\mathbf{x} \in K$. Note that (1.3) is equivalent to

$$M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'_+(\mathbf{x}) \preceq -2\nu M(\mathbf{x}),$$

where $A \preceq B$ for $A, B \in \mathbb{S}^{n\times n}$ means $A - B$ is negative semi-definite, i.e. $\mathbf{w}^T(A-B)\mathbf{w} \leq 0$ for all $\mathbf{w} \in \mathbb{R}^n$, see [19, Remark 2.5].

The following theorem outlines how contraction metrics can be used to locate equilibria and their basins of attraction.

1.6 Theorem (Existence and uniqueness of the equilibrium) *[19, Theorem 3.1]*
Let $\emptyset \neq K \subset \mathbb{R}^n$ be a compact, connected, and positively invariant set and M be a Riemannian metric defined on a neighborhood G of K and contracting in K with exponent $-\nu < 0$ as in Definition 1.4. Then there exists one and only one equilibrium $\mathbf{x}_0$ of system (1.1) *in K; $\mathbf{x}_0$ is exponentially stable and K is a subset of its basin of attraction $\mathcal{A}(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{R}^n : \lim_{t\to\infty} \boldsymbol{\phi}(t,\mathbf{x}) = \mathbf{x}_0\}$.*

The following theorem, which is a converse statement to Theorem 1.6, guarantees that within the basin of attraction for an exponentially stable equilibrium of (1.1) there exists a contraction metric. Note that it provides a stronger smoothness property for M than required in Definition 1.1 and thus, it allows us to use the orbital derivative instead of the forward orbital derivative (see Remark 1.2). Note that on a compact subset $K \subset \mathcal{A}(\mathbf{x}_0)$, M is a contraction metric by Remark 1.5.

1.7 Theorem (Existence of contraction metrics) *[32, Theorem 2.2, Theorem 2.3]*
Let $\mathbf{f} \in C^s(\mathbb{R}^n;\mathbb{R}^n)$, $s \geq 2$. Let $\mathbf{x}_0$ be an exponentially stable equilibrium of (1.1) *with basin*

of attraction $\mathcal{A}(\mathbf{x}_0)$. Let $C \in C^{s-1}(\mathcal{A}(\mathbf{x}_0); \mathbb{S}^{n\times n})$ be such that $C(\mathbf{x})$ is a positive definite matrix for all $\mathbf{x} \in \mathcal{A}(\mathbf{x}_0)$. Then the matrix PDE

$$F(M)(\mathbf{x}) := M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) + M'(\mathbf{x}) = -C(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \mathcal{A}(\mathbf{x}_0) \tag{1.4}$$

has a unique solution.

In particular, $M \in C^{s-1}(\mathcal{A}(\mathbf{x}_0); \mathbb{S}^{n\times n})$, $M(\mathbf{x})$ is positive definite for all $\mathbf{x} \in \mathcal{A}(\mathbf{x}_0)$, and M is of the form

$$M(\mathbf{x}) = \int_0^\infty \psi(\tau, \mathbf{x})^T C(\boldsymbol{\phi}(\tau, \mathbf{x}))\psi(\tau, \mathbf{x})\, d\tau, \tag{1.5}$$

where $\tau \mapsto \boldsymbol{\phi}(\tau, \mathbf{x})$ is the solution to (1.1) with initial value $\boldsymbol{\phi}(0, \mathbf{x}) = \mathbf{x}$ and $\tau \mapsto \psi(\tau, \mathbf{x})$ is the principal fundamental matrix solution to $\dot{Y} = D\mathbf{f}(\boldsymbol{\phi}(t, \mathbf{x}))Y$; i.e. $Y(0) = I$.

The integral formula (1.5) is the starting point for our numerical computation of a contraction metric. The idea is to numerically approximate $M(\mathbf{x})$, using (1.5), by $\widetilde{M}(\mathbf{x})$ for all $\mathbf{x}$ in a finite set $X \subset \mathbb{R}^n$. Then we will interpolate the values $\widetilde{M}(\mathbf{x})$ to obtain a CPA function, for which we can then rigorously check the conditions of Theorem 1.6.

1.2 Estimation of $M(\boldsymbol{\xi})$

The matrix valued function M in formula (1.5) can be estimated arbitrarily close at a point $\boldsymbol{\xi} \in \mathcal{A}(\mathbf{x}_0$ by using numerical integration and numerical quadrature. To this end, we first fix a matrix-valued function $C \in C^{s-1}(\mathbb{R}^n; \mathbb{S}^{n\times n})$, which in practice can be taken simply as the constant identity matrix $I \in \mathbb{R}^{n\times n}$, a time horizon $H > 0$, and a set of points X, at which we compute values for our approximate metric $\widetilde{M}$ inspired by (1.5). For $\boldsymbol{\xi} \in X$ we first compute a numerical solution $\widetilde{\boldsymbol{\phi}} : [0, H] \to \mathbb{R}^n$ to the initial-value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \boldsymbol{\xi}, \tag{1.6}$$

on the time horizon $[0, H]$. We do this by fixing the number of time steps N and the corresponding length of a uniform time step $h := H/N$ and then generate a sequence of vectors $\widetilde{\boldsymbol{\phi}}_i$, $i = 0, 1, \ldots, N$, such that $\widetilde{\boldsymbol{\phi}}_i \approx \boldsymbol{\phi}(t_i, \boldsymbol{\xi})$ approximates the true solution $\boldsymbol{\phi}(\cdot, \boldsymbol{\xi})$ to the initial-value problem (1.6) at time $t_i := ih$. In the sequel $\widetilde{\boldsymbol{\phi}} = \widetilde{\boldsymbol{\phi}}(\cdot, \boldsymbol{\xi})$ refers to the function defined by linearly interpolating these values.

Then we use our approximate solution $\widetilde{\boldsymbol{\phi}}$ to (1.6) to obtain an approximation $\widetilde{Y}$ of the principal fundamental matrix solution to $\dot{Y} = D\mathbf{f}(\boldsymbol{\phi}(t, \boldsymbol{\xi}))Y$. That is, we solve numerically the matrix-valued initial-value problem

$$\dot{Y} = g(t)Y, \quad Y(0) = I, \tag{1.7}$$

where $D\mathbf{f}(\boldsymbol{\phi}(t, \boldsymbol{\xi}))$ has been substituted by the approximation $g(t) := D\mathbf{f}(\widetilde{\boldsymbol{\phi}}(t, \boldsymbol{\xi}))$. In practice, we use the Adams-Bashforth method of order 4 (AB4) initialized with the usual Runge-Kutta method of order 4 (RK4) to numerically solve (1.6) and (1.7) at times $t_i = ih$ to obtain the approximation $\widetilde{Y}_i$.

The algorithm to approximate the solutions to the initial-value problems (1.6) and (1.7) can now be summarized as:

1. Fix the time horizon H and the number of time steps N.

2. For the initiation phase for the AB4 multi-step method, fix $\widetilde{\mathbf{x}}_0 = \boldsymbol{\xi}$ and set $h_0 = \frac{1}{2}H/N$.

3. Use RK4 with $h = h_0$ to compute $\widetilde{\mathbf{x}}_{i+1}$ for $i = 0, 1, 2, 3, 4, 5$.

4. Relabel the solution terms using $\widetilde{\boldsymbol{\phi}}_{i/2} = \widetilde{\mathbf{x}}_i$ for $i = 0, 1, \ldots, 6$, e.g. $\widetilde{\boldsymbol{\phi}}_{\frac{1}{2}} = \widetilde{\mathbf{x}}_1$, $\widetilde{\boldsymbol{\phi}}_1 = \widetilde{\mathbf{x}}_2$ etc.

5. Set $\widetilde{Y}_0 = I$, $h = H/N$, and use RK4 to compute $\widetilde{Y}_{i+1}$ for $i = 0, 1, 2$.

6. Now the initialization phase is over and we have $\widetilde{\boldsymbol{\phi}}_i$ and $\widetilde{Y}_i$ at our disposal for $i = 0, 1, 2, 3$. Set $h = H/N$ and use the AB4 for $i = 3, 4, \ldots, N-1$ to compute the remaining $\widetilde{\boldsymbol{\phi}}_i$ and $\widetilde{Y}_i$ for $i = 4, 5, \ldots, N$.

Finally, we compute an approximation $\widetilde{M}(\boldsymbol{\xi})$ to the integral

$$\int_0^H Y^T(\tau)\, C\left(\boldsymbol{\phi}(\tau, \boldsymbol{\xi})\right) Y(\tau) d\tau \approx \widetilde{M}(\boldsymbol{\xi}); \tag{1.8}$$

note that there are several approximations to the actual integral: firstly, we use a Romberg-like numerical quadrature to compute the integral, for which we now only need values of the integrand at discrete time steps; secondly, we replace the values of the integrand with our numerical solutions $\widetilde{\boldsymbol{\phi}}_i$ and $\widetilde{Y}_i$ to (1.6) and (1.7), respectively.

In more detail, with $N = 2^p q$, $p, q \in \mathbb{N}_+$, the approximation $\widetilde{R}_{r,0}$ to the integral is computed by using the composite Trapezoidal rule with $N/2^r$ intervals, $r = 0, 1, 2, \ldots, p$. Then for $1 \le r + s \le p$ the numbers $\widetilde{R}_{r,s}$ are obtained by (Romberg) extrapolation.

1. Define recursively $N_0 := N$ and $N_{k+1} := N_k/2$ for $k = 0, \ldots, p-1$; note that $N_p = q$. Define $h'_k := H/N_k$ for $k = 0, \ldots, p$. Set

$$\mathrm{Trap}_k := h'_k \left(\frac{\widetilde{\alpha}_0 + \widetilde{\alpha}_{N_R}}{2} + \sum_{j=1}^{N_k - 1} \alpha_{j2^k} \right) \quad \text{for} \quad k = 0, 1, \ldots, p, \tag{1.9}$$

where $\widetilde{\alpha}_i := \widetilde{Y}_i^T C(\widetilde{\boldsymbol{\phi}}_i)\widetilde{Y}_i$

2. Extrapolate by using the tableau

$$\widetilde{R}_{r,0} := \mathrm{Trap}_r \quad \text{for} \quad r = 0, 1, \ldots, p \tag{1.10}$$

and then for $s = 1, \ldots, p$,

$$\widetilde{R}_{r,s} := \frac{4^s \widetilde{R}_{r,s-1} - \widetilde{R}_{r+1,s-1}}{4^s - 1} \quad \text{for} \quad 0 \le r \le p - s. \tag{1.11}$$

We set $\widetilde{M}(\boldsymbol{\xi}) := \widetilde{R}_{0,p}$. We use the same number of time steps N and hence use the uniform time step $h = H/N$ from the AB4 method. Note that for the initialization with the Runge-Kutta method for (1.7) we require the time step $h_0 = h/2$, as explained above.

In the following, we need to fix a compact set $S \subset \mathbb{R}^n$ which is positively invariant, both for the solution $\boldsymbol{\phi}$ and its numerical approximation $\widetilde{\boldsymbol{\phi}}$, so that we have a Lipschitz constant for any locally Lipschitz continuous function in S, in particular for continuously differentiable functions on S. Level sets of numerically computed Lyapunov-like functions can be used for identifying such sets, cf. [31, Theorems 3.2, 3.5].

The following theorem, delivers uniform error estimates on compact sets $K \subset \mathcal{A}(\mathbf{x}_0)$ for $\widetilde{M}$ computed as described above using large enough parameters H and N in the computations; for a detailed explanation of the method and a proof of the theorem see [30].

1.8 Theorem (Error estimate) *[30, Theorem 4.1]*
Assume $\mathbf{f}$ *in* (1.1) *is* $C^{2(p+1)+1}$ *for* $p \in \mathbb{N}_+$ *and let* M *be the solution to the PDE* (1.4) *on* $\mathcal{A}(\mathbf{x}_0)$*, whose existence is guaranteed by Theorem 1.7, and is given by formula* (1.5)

$$M(\boldsymbol{\xi}) = \int_0^\infty \psi(\tau, \boldsymbol{\xi})^T C(\boldsymbol{\phi}(\tau, \boldsymbol{\xi}))\psi(\tau, \boldsymbol{\xi})\, d\tau.$$

Then, for any compact $K \subset \mathcal{A}(\mathbf{x}_0)$ *and* $\varepsilon^* > 0$*, there exists* $H^* > 0$ *such that for all fixed and finite* $H \geq H^*$ *there exist an* $N^* = 2^p q^*$*, where* $q^* \in \mathbb{N}_+$*, such that for all* $N = 2^p q$*,* $q \in \mathbb{N}_+$ *and* $q \geq q^*$*, we have*

$$\|M(\boldsymbol{\xi}) - \widetilde{M}(\boldsymbol{\xi})\|_2 \leq \varepsilon^*$$

for all $\boldsymbol{\xi} \in K$*. Here* $\widetilde{M}(\boldsymbol{\xi})$ *is the Romberg extrapolation* $\widetilde{R}_{0,p}$ *of the integral* (1.8) *using the interval* H *and the numerical solutions* $\widetilde{\boldsymbol{\phi}}_i$ *and* $\widetilde{Y}_i$ *to* (1.6) *and* (1.7) *in the integrand and the step-size* $h = H/N$ *in both the numerical integration and quadrature.*

1.9 Remark We use the $\|\cdot\|_2$-norm rather than the $\|\cdot\|_{\max}$-norm used in [30, Theorem 4.1]; the equivalence is obvious from the equivalence of norms on $\mathbb{R}^n$.

Note that $\mathbf{f} \in C^{2(p+1)+1}(\mathbb{R}^n; \mathbb{R}^n)$ and $N = 2^p q$ for $p, q \in \mathbb{N}_+$ enables the use of p subsequent Romberg like extrapolations for the numerical quadrature to obtain an $O(h^{2(p+1)})$ method. Since the AB4 and RK4 methods are of order $O(h^4)$ not much is gained by having $p > 1$, but on the other hand the computations using a larger p are very cheap; see [30, §3.3] for the details. Further, for a smooth enough $\mathbf{f}$, one could use a numerical scheme of higher order to solve the the initial value problems and the numerical quadrature would conserve this higher order of the approximation. Further note that we are implicitly assuming that $C \in C^{2(p+1)}(\mathcal{A}(\mathbf{x}_0); \mathbb{S}^{n\times n})$ in the PDE (1.4), cf. Theorem 1.7.

Note that H^* and N^* depend on $K \subset \mathcal{A}(\mathbf{x}_0)$ and $\varepsilon^* > 0$, but not on the particular $\boldsymbol{\xi} \in K$.

2 CPA Interpolation

In the previous section we explained how one can estimate numerically the values of a certain contraction metric for the system (1.1) at discrete points. In this section we discuss how to interpolate such values and verify that the interpolation is a contraction metric for the system. As a first step the domain of interest is triangulated. We do this using the so-called standard triangulation from [22, Def. 13] in $\mathbb{R}^n_+$, translated and scaled, which has many nice properties and can be implemented efficiently [33]. However, more general triangulations can be used, see e.g. [3].

The goal is to interpolate $\widetilde{M}$ by a continuous piecewise affine function P, for which we can verify that P is a contraction metric, i.e. $P(\mathbf{x})$ is positive definite and $F(P)(\mathbf{x})$ is negative definite, see (1.4) for the definition. Since P is affine on each simplex, we can use Taylor-based estimates to achieve this goal.

First we explain how the CPA interpolation is defined. For this we need to fix a triangulation, i.e. a set of simplices satisfying certain properties. Recall that an n-simplex $\mathfrak{S}$ in $\mathbb{R}^n$ is the convex hull

$$\mathfrak{S} = \mathrm{co}(\mathbf{x}_0, \ldots, \mathbf{x}_n) = \left\{ \sum_{k=0}^{n} \lambda_k \mathbf{x}_k \,:\, \lambda_k \in [0,1] \text{ and } \sum_{k=0}^{n} \lambda_k = 1 \right\}$$

of affinely independent vectors $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^n$ called its vertices, and a face of $\mathfrak{S}$ is the convex hull of a nonempty subset of its vertices. For our discussion it is advantageous to have the order of the vertices of a simplex fixed, but the results do not depend on this ordering.

2.1 Definition (Triangulation) *We call a set $\mathcal{T} = \{\mathfrak{S}_\nu\}_\nu$ of n-simplices $\mathfrak{S}_\nu$ a triangulation in $\mathbb{R}^n$, if two simplices $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$, $\mu \neq \nu$, intersect in a common face or not at all. The set $\mathcal{T}$ can be infinite, but for every compact set $K \subset \mathbb{R}^n$ we demand that the set $\{\mathfrak{S}_\nu \cap K : \mathfrak{S}_\nu \in \mathcal{T}\}$ is finite. For a triangulation $\mathcal{T}$ we define its* domain *and* vertex set *as*

$$\mathcal{D}_\mathcal{T} := \bigcup_\nu \mathfrak{S}_\nu$$

and

$$\mathcal{V}_\mathcal{T} := \{\mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{x} \text{ is a vertex of a simplex in } \mathcal{T}\ \}.$$

We also say that $\mathcal{T}$ is a triangulation of the set $\mathcal{D}_\mathcal{T}$.

*For a triangulation $\mathcal{T} = \{\mathfrak{S}_\nu\}_\nu$ and constants $h, b > 0$, we say that $\mathcal{T}$ is (h,b)-*bounded *if it fulfills the following conditions:*

(i) The diameter of every simplex $\mathfrak{S}_\nu \in \mathcal{T}$ is bounded by h, that is

$$h_\nu := \mathrm{diam}(\mathfrak{S}_\nu) := \max_{\mathbf{x},\mathbf{y} \in \mathfrak{S}_\nu} \|\mathbf{x} - \mathbf{y}\|_2 < h.$$

(ii) The degeneracy of every simplex $\mathfrak{S}_\nu \in \mathcal{T}$ is bounded by b in the sense that

$$h_\nu \|X_\nu^{-1}\|_1 \leq b,$$

where $X_\nu := (\mathbf{x}_1^\nu - \mathbf{x}_0^\nu, \mathbf{x}_2^\nu - \mathbf{x}_0^\nu, \ldots, \mathbf{x}_n^\nu - \mathbf{x}_0^\nu)^T$ is the so-called shape matrix *of the simplex $\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)$.*

*If additionally the diameters of the simplices satisfy $h \leq \ell h_\nu$ for all $\mathfrak{S}_\nu$ with a constant $\ell \geq 1$, then we say that the triangulation is (h,b,ℓ)-*bounded. *In this case we additionally have the estimate*

$$\frac{h\|X_\nu^{-1}\|_1}{\ell} \leq h_\nu \|X_\nu^{-1}\|_1 \leq b, \quad \text{i.e.} \quad X_\nu^{-1}\|_1 \leq \frac{\ell b}{h} \tag{2.1}$$

2.2 Remark Note that all triangulations that are finite sets of n-simplices are automatically (h,b,ℓ)-bounded for some constants $h, b, \ell > 0$. This concept is mainly useful when studying sequences of infinite triangulations and their intersections with compact sets. In essence, one can use an infinite triangulation $\mathcal{T}^* = \{\mathfrak{S}_\nu\}_\nu$ that is (h,b,ℓ)-bounded and take advantage of the fact that for any scaling-parameter $\sigma > 0$ the triangulation $\mathcal{T} := \{\sigma\mathfrak{S}_\nu : \mathfrak{S}_\nu \in \mathcal{T}^*, (\sigma\mathfrak{S}_\nu)\cap\mathcal{D}^\circ \neq \emptyset\}$ is also $(\sigma h, b, \ell)$-bounded for a compact $\mathcal{D} \subset \mathbb{R}^n$ and obviously $\mathcal{D}_\mathcal{T} \supset \mathcal{D}$. Further, the existence of a finite constant $b > 0$ is independent of the order of the vertices (for h and ℓ this is obvious). For a detailed discussion of this see [25]; note in particular that for any compact $\mathcal{D} \subset \mathbb{R}^n$ and any $h > 0$, one can scale down the so-called standard triangulation to obtain an $(h, 2\sqrt{n}, 1)$-bounded triangulation covering $\mathcal{D}$.

2.3 Definition (CPA interpolation) *Let $\mathcal{T}$ be a triangulation in $\mathbb{R}^n$ and assume some values $\widetilde{M}_{ij}(\mathbf{x}_k) \in \mathbb{R}$ are fixed for every $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$ and every $i, j = 1, 2, \ldots, n$. Then we can uniquely define continuous functions $P_{ij} : \mathcal{D}_\mathcal{T} \to \mathbb{R}$ through:*

(i) $P_{ij}(\mathbf{x}) := \widetilde{M}_{ij}(\mathbf{x})$ for every $\mathbf{x} \in \mathcal{V}_\mathcal{T}$,

(ii) P_{ij} is affine on every simplex $\mathfrak{S}_\nu \in \mathcal{T}$, i.e. there is a vector $\mathbf{w}_{ij}^\nu \in \mathbb{R}^n$ and a number $b_{ij}^\nu \in \mathbb{R}$, such that

$$P_{ij}(\mathbf{x}) = (\mathbf{w}_{ij}^\nu)^T\mathbf{x} + b_{ij}^\nu$$

for all $\mathbf{x} \in \mathfrak{S}_\nu$.

With $P(\mathbf{x}) := (P_{ij}(\mathbf{x}))_{i,j=1,2,\ldots,n}$ we obtain a matrix valued function $P\colon \mathcal{D}_\mathcal{T} \to \mathbb{R}^{n\times n}$ from the matrices $\widetilde{M}(\mathbf{x}_k) := (\widetilde{M}_{ij}(\mathbf{x}_k))_{i,j=1,2,\ldots,n}$, $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$. Further, if $\widetilde{M}(\mathbf{x}_k) \in \mathbb{S}^{n\times n}$ for every $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$, then $P\colon \mathcal{D}_\mathcal{T} \to \mathbb{S}^{n\times n}$.

For every simplex $\mathfrak{S}_\nu \in \mathcal{T}$ we define $\nabla P_{ij}^\nu := \nabla P_{ij}|_{\mathfrak{S}_\nu^\circ} = \mathbf{w}_{ij}^\nu$.

2.4 Remark (Orbital derivative) Let $P(\mathbf{x})$ be as in Definition 2.3, fix a point $\mathbf{x} \in \mathcal{D}_\mathcal{T}^\circ$, and consider system (1.1). As shown in the proof of [21, Lemma 4.7], there exists a $\mathfrak{S}_\nu = \mathrm{co}(\mathbf{x}_0, \ldots, \mathbf{x}_n) \in \mathcal{T}$ and a number $\theta^* > 0$ such that $\mathbf{x} + \theta\mathbf{f}(\mathbf{x}) \in \mathfrak{S}_\nu$ for all $\theta \in [0, \theta^*]$. Then the forward orbital derivative $(P_{ij})'_+(\mathbf{x})$, defined by formula (1.2) (see Remark 1.3), is given by

$$(P_{ij})'_+(\mathbf{x}) = \nabla P_{ij}^\nu \cdot \mathbf{f}(\mathbf{x}),$$

where ∇P_{ij}^ν was defined in Definition 2.3. Thus

$$P'_+(\mathbf{x}) = P(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T P(\mathbf{x}) + \left(\nabla P_{ij}^\nu \cdot \mathbf{f}(\mathbf{x})\right)_{i,j=1,2,\ldots,n}.$$

For our purposes, however, this is not sufficient, because, for an $\mathbf{x} \in \mathcal{D}_\mathcal{T}$ that is in more than one simplex, we do not want to check which gradient should be used for the orbital derivative. This is the motivation for the definition of $A_\nu(\mathbf{x})$ for *all* ν such that $\mathbf{x} \in \mathfrak{S}_\nu$ in the next definition.

2.5 Definition *Let $P\colon \mathcal{D}_\mathcal{T} \to \mathbb{S}^{n\times n}$ be as in Definition 2.3. Then, for every $\mathbf{x} \in \mathcal{D}_\mathcal{T}$ and every ν such that $\mathbf{x} \in \mathfrak{S}_\nu$, we define*

$$A_\nu(\mathbf{x}) := P(\mathbf{x})D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T P(\mathbf{x}) + \left(\nabla P_{ij}^\nu \cdot \mathbf{f}(\mathbf{x})\right)_{i,j=1,2,\ldots,n}, \tag{2.2}$$

with respect to the system (1.1). Note that $(\nabla P_{ij}^\nu \cdot \mathbf{f}(\mathbf{x}_k))_{i,j=1,2,\ldots,n}$ is the symmetric $(n\times n)$-matrix with entries $\nabla P_{ij}^\nu \cdot \mathbf{f}(\mathbf{x})$, where ∇P_{ij}^ν is defined as in Definition 2.3; in particular ∇P_{ij}^ν is a fixed vector for a given $\mathfrak{S}_\nu$ and i, j and is independent of $\mathbf{x}$.

Note that if $\mathbf{x} \in \mathfrak{S}_\nu^\circ$, then necessarily $P'_+(\mathbf{x}) = A_\nu(\mathbf{x})$ and there are no further simplices $\mathfrak{S}_\mu \in \mathcal{T}$ such that $\mathbf{x} \in \mathfrak{S}_\mu$. In general, however, we have for an $\mathbf{x} \in \mathcal{D}_\mathcal{T}^\circ$ that $P'_+(\mathbf{x}) = A_\nu(\mathbf{x})$ for *some* ν such that $\mathbf{x} \in \mathfrak{S}_\nu$. In the Verification Problem 2.6 below, where we make constraints at the vertices of the triangulation $\mathcal{T}$, and a vertex is in more than one simplex except for the trivial case that $\mathcal{T}$ consist of exactly one simplex, this problem is tackled by making constraints for all possibilities.

A CPA interpolation of the values $\widetilde{M}(\mathbf{x}_k)$, $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$, that satisfies the following constraints in the Verification Problem 2.6 is a contraction metric, see Theorem 2.7. The converse statement is true as well, see Theorem 2.11: if P is a contraction metric and the triangulation is sufficiently fine and not degenerate, then the CPA interpolation will satisfy the constraints of the verification problem.

2.6 Verification Problem *Given is a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} \in C^3(\mathbb{R}^n; \mathbb{R}^n)$, a triangulation $\mathcal{T}$ in $\mathbb{R}^n$, and the values $P(\mathbf{x}_k) \in \mathbb{S}^{n\times n}$ at all vertices $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$. The verification problem consists of the following constants and inequalities:*

Constants: *The constants of the verification problem, fixed or computed, are:*

1. $\varepsilon_0 > 0$ *– lower bound (to enforce strict positive/negative definiteness).*
2. *The diameter h_ν of each simplex $\mathfrak{S}_\nu \in \mathcal{T}$:*
$$h_\nu := \operatorname{diam}(\mathfrak{S}_\nu) = \max_{\mathbf{x},\mathbf{y} \in \mathfrak{S}_\nu} \|\mathbf{x} - \mathbf{y}\|_2.$$
3. *Upper bounds $B_{2,\nu}$ on the second-order derivatives of the components f_k of $\mathbf{f}$ on each simplex $\mathfrak{S}_\nu \in \mathcal{T}$:*
$$B_{2,\nu} \geq \max_{\substack{\mathbf{x}\in\mathfrak{S}_\nu \\ i,j,k=1,2,\ldots,n}} \left| \frac{\partial^2 f_k}{\partial x_i \partial x_j}(\mathbf{x}) \right|.$$
4. *Upper bounds $B_{3,\nu}$ on the third-order derivatives of the components f_l of $\mathbf{f}$ on each simplex $\mathfrak{S}_\nu \in \mathcal{T}$:*
$$B_{3,\nu} \geq \max_{\substack{\mathbf{x}\in\mathfrak{S}_\nu \\ i,j,k,l=1,\ldots,n}} \left| \frac{\partial^3 f_l}{\partial x_i \partial x_j \partial x_k}(\mathbf{x}) \right|.$$

Verification: *We check whether the following inequalities hold:*

(VP1) **Positive definiteness of P**

For each $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$:
$$P(\mathbf{x}_k) \succeq \varepsilon_0 I.$$

(VP2) **Negative definiteness of $\mathbf{A}_\nu$**

For each simplex $\mathfrak{S}_\nu = \operatorname{co}(\mathbf{x}_0, \ldots, \mathbf{x}_n) \in \mathcal{T}$ and each vertex $\mathbf{x}_k$ of $\mathfrak{S}_\nu$:
$$-\varepsilon_0 I \succeq A_\nu(\mathbf{x}_k) + h_\nu^2 E_\nu I.$$

Here $A_\nu(\mathbf{x}_k)$ is as in (2.2) and
$$E_\nu := n^2(1 + 4\sqrt{n}) B_{2,\nu} D_\nu + 2\, n^3 B_{3,\nu} C_\nu, \tag{2.3}$$

where $C_\nu, D_\nu \geq 0$ are constants fulfilling

$$P(\mathbf{x}_k) \preceq C_\nu I$$

for all vertices $\mathbf{x}_k$ of $\mathfrak{S}_\nu$ and

$$\left\|\nabla P_{ij}^\nu\right\|_1 \leq D_\nu$$

for all $i, j = 1, 2, \ldots, n$.

2.7 Theorem (CPA contraction metric) *[27, Theorem 4.9]* *Let $\mathbf{f} \in C^3(\mathbb{R}^n, \mathbb{R}^n)$. Assume the conditions of the Verification Problem 2.6 are satisfied. Then the matrix-valued function P, where $P(\mathbf{x})$ is interpolated from the values $P_{ij}(\mathbf{x}_k)$ as in Definition 2.3, is a Riemannian metric contracting in any compact set $K \subset \mathcal{D}_{\mathcal{T}}^\circ$.*

2.8 Remark The following points are worth noting.

1. The gradient ∇P_{ij}^ν of the affine function $P_{ij}\big|_{\mathfrak{S}_\nu}$ on the simplex $\mathfrak{S}_\nu = \operatorname{co}(\mathbf{x}_0, \ldots, \mathbf{x}_n)$ can be computed by the expression

$$\nabla P_{ij}^\nu := X_\nu^{-1} \begin{pmatrix} P_{ij}(\mathbf{x}_1) - P_{ij}(\mathbf{x}_0) \\ \vdots \\ P_{ij}(\mathbf{x}_n) - P_{ij}(\mathbf{x}_0) \end{pmatrix} \in \mathbb{R}^n, \tag{2.4}$$

   where $X_\nu = (\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \ldots, \mathbf{x}_n - \mathbf{x}_0)^T \in \mathbb{R}^{n \times n}$ is the shape-matrix of the simplex $\mathfrak{S}_\nu$.

2. The Verification Problem 2.6 can also be solved as a semi-definite feasibility problem. In this case the matrices $P(\mathbf{x}_k)$, $\mathbf{x}_k \in \mathcal{V}_{\mathcal{T}}$, are not fixed a priori but are variables of the problem. Further, the C_ν are not constants but variables and the D_ν and the conditions $\|\nabla P_{ij}^\nu\|_1 \leq D_\nu$ are implemented using the auxiliary variables D_ν^k and the constraints

$$-D_\nu^k \leq [\nabla P_{ij}^\nu]_k \leq D_\nu^k \quad \text{for } k = 1, \ldots, n,$$

   where $[\nabla P_{ij}^\nu]_k$ is the k-th component of the vector ∇P_{ij}^ν, which is a vector of linear combinations of variables, cf. (2.4). Then one sets $D_\nu := \sum_{k=1}^n D_\nu^k$. Hence, a feasible solution to Verification Problem 2.6 delivers a symmetric matrix $P(\mathbf{x}_k) = (P_{ij}(\mathbf{x}_k))_{i,j=1,2,\ldots,n}$ at each vertex $\mathbf{x}_k$ of the triangulation $\mathcal{T}$ and values C_ν and D_ν for each simplex $\mathfrak{S}_\nu \in \mathcal{T}$.

3. It is much more efficient to verify for a given set of matrices $P(\mathbf{x}_k)$, $\mathbf{x}_k \in \mathcal{V}_{\mathcal{T}}$, whether they constitute a feasible solution to the Verification Problem 2.6 or not, than it is to actually solve it.

The following theorem provides an error estimate for the CPA interpolation P of the numerical approximation $\widetilde{M}$ to M.

2.9 Theorem (Error estimate for the CPA interpolation of the contraction metric) *Assume that $\mathbf{x}_0$ is an exponentially stable equilibrium of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f} \in C^{2(p+1)+1}(\mathbb{R}^n; \mathbb{R}^n)$, with $p \in \mathbb{N}_+$. Let $\Omega \subset \mathcal{A}(\mathbf{x}_0)$ be a bounded open set with C^1 boundary and $K \subset \Omega$ be a positively invariant and compact set. Let $M \in C^{2(p+1)+1}(\mathcal{A}(\mathbf{x}_0); \mathbb{S}^{n \times n})$*

be the solution of the PDE (1.4) from Theorem 1.7 for some $C \in C^{2(p+1)}(\mathcal{A}(\mathbf{x}_0);\mathbb{S}^{n\times n})$, $C(\mathbf{x}) \succ 0$ for all $\mathbf{x} \in \mathcal{A}(\mathbf{x}_0)$. Let $\varepsilon^* > 0$ and $H^* > 0$ be given as in Theorem 1.8. That is, for all $\boldsymbol{\xi} \in K$, we have

$$\|M(\boldsymbol{\xi}) - \widetilde{M}(\boldsymbol{\xi})\|_2 \le \varepsilon^*,$$

where $\widetilde{M}(\boldsymbol{\xi})$ is computed using $H \ge H^*$ and $N = 2^p q$, $q \ge q^*$, where $N^* = 2^p q^*$, $q^* \in \mathbb{N}_+$, is a suitable constant for H. In particular the time-step in the methods to compute $\widetilde{M}(\boldsymbol{\xi})$ is $h = H/N$.

Furthermore, let P be the CPA interpolation of $\widetilde{M}$ on an $(h_{tri.}, b, \ell)$-bounded triangulation $\mathcal{T} = \{\mathfrak{S}_\nu\}_\nu$ with $\mathcal{D}_\mathcal{T} \subset K$. That is, for every vertex $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$ of the triangulation $\mathcal{T}$ we set $P(\mathbf{x}_k) = \widetilde{M}(\mathbf{x}_k)$ and then we interpolate these values over $\mathcal{D}_\mathcal{T}$ as in Definition 2.3.

Then, the following error estimates hold true for all $\mathbf{x} \in \mathcal{D}_\mathcal{T}$:

$$\|M(\mathbf{x}) - P(\mathbf{x})\|_2 \le n\, h_{tri.}^2 B_M + \varepsilon^*, \tag{2.5}$$

For all ν such that $\mathbf{x} \in \mathfrak{S}_\nu \in \mathcal{T}$ we have

$$\|F(M)(\mathbf{x}) - A_\nu(\mathbf{x})\|_2 \le \left[(\gamma + 2h_{tri.})n h_{tri.} B_M + 2\left(\frac{\ell b n^3}{h_{tri.}} + 1\right)\varepsilon^*\right] B_{\mathbf{f}}, \tag{2.6}$$

where $\gamma := 1 + \dfrac{bn^{3/2}}{2}$, $B_M := \|M\|_{C^2(\Omega;\mathbb{S}^{n\times n})}$, and $B_{\mathbf{f}} := \|\mathbf{f}\|_{C^1(\Omega;\mathbb{R}^n)}$.

Proof: Denote by M_c the CPA interpolation of M on $\mathcal{T}$. First, note that by [27, Lemma 4.5] we have for an arbitrary $\mathbf{x} \in \mathcal{D}_\mathcal{T}$ that

$$\begin{aligned}\|M(\mathbf{x}) - P(\mathbf{x})\|_2 &\le \|M(\mathbf{x}) - M_c(\mathbf{x})\|_2 + \|M_c(\mathbf{x}) - P(\mathbf{x})\|_2 \\ &\le n\, h_{\text{tri.}}^2 \|M\|_{C^2(\Omega;\mathbb{S}^{n\times n})} + \|M_c(\mathbf{x}) - P(\mathbf{x})\|_2 .\end{aligned}$$

In order to provide an estimate for the latter term choose a simplex $\mathfrak{S}_\nu$ such that $\mathbf{x} \in \mathfrak{S}_\nu = \text{co}(\mathbf{x}_0, \ldots, \mathbf{x}_n)$. Then $\mathbf{x}$ can be written uniquely as $\mathbf{x} = \sum_{k=0}^n \lambda_k \mathbf{x}_k$ with $\lambda_k \in [0,1]$ and $\sum_{k=0}^n \lambda_k = 1$. Thus, we obtain

$$\begin{aligned}\|M_c(\mathbf{x}) - P(\mathbf{x})\|_2 &= \left\| M_c\left(\sum_{k=0}^n \lambda_k \mathbf{x}_k\right) - P\left(\sum_{k=0}^n \lambda_k \mathbf{x}_k\right)\right\|_2 \\ &= \left\|\sum_{k=0}^n \lambda_k \big(M_c(\mathbf{x}_k) - P(\mathbf{x}_k)\big)\right\|_2 \\ &\le \sum_{k=0}^n \lambda_k \left\| M(\mathbf{x}_k) - \widetilde{M}(\mathbf{x}_k)\right\|_2 \\ &\le \varepsilon^*.\end{aligned} \tag{2.7}$$

We have shown the estimate (2.5).

To prove the estimate (2.6), let again $\mathbf{x} \in \mathcal{D}_\mathcal{T}$ be arbitrary but fixed and let $\mathfrak{S}_\nu \in \mathcal{T}$ be any simplex such that $\mathbf{x} \in \mathfrak{S}_\nu$. We consider

$$\begin{aligned}&F(M)(\mathbf{x}) - A_\nu(\mathbf{x}) \\ &\quad = \big(M(\mathbf{x}) - P(\mathbf{x})\big)D\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T\big(M(\mathbf{x}) - P(\mathbf{x})\big) + M'(\mathbf{x}) - \big(\nabla P_{ij}^\nu \cdot \mathbf{f}(\mathbf{x})\big)_{i,j=1,2,\ldots,n} .\end{aligned} \tag{2.8}$$

Observe that

$$\left\|D\mathbf{f}(\mathbf{x})^T\right\|_2 = \|D\mathbf{f}(\mathbf{x})\|_2 \leq \|\mathbf{f}\|_{C^1(\Omega;\mathbb{R}^n)} = B_{\mathbf{f}}$$

by inequality (A.6) of [27, Remark A.1] and we get by (2.5) that

$$\|M(\mathbf{x}) - P(\mathbf{x})\|_2 \|D\mathbf{f}(\mathbf{x})\|_2 \leq \left(n\, h_{\text{tri.}}^2 B_M + \varepsilon^*\right) B_{\mathbf{f}}.$$

This takes care of the first two terms on the right-hand side of (2.8). For the last term, recall from [27, Lemma 4.5] that for $i, j = 1, 2, \ldots, n$ we have

$$\left\|\nabla M_{ij}(\mathbf{x}) - \nabla (M_c)_{ij}\big|_{\mathfrak{S}_\nu^\circ}\right\|_1 \leq h_{\text{tri.}} \left(1 + \frac{bn^{3/2}}{2}\right) \|M\|_{C^2(\Omega;\mathbb{S}^{n\times n})} = h_{\text{tri.}}\gamma B_M.$$

Now, additionally using Hölder inequality, and $\|\mathbf{f}\|_{C^0(\Omega;\mathbb{R}^n)} \leq \|\mathbf{f}\|_{C^1(\Omega;\mathbb{R}^n)} = B_{\mathbf{f}}$, we have

$$\begin{aligned}
\left\|M'(\mathbf{x}) - \left(\nabla P_{ij}^\nu \cdot \mathbf{f}(\mathbf{x})\right)_{i,j=1,2,\ldots,n}\right\|_2 &\leq n \left\|\left(\left[\nabla M_{ij}(\mathbf{x}) - \nabla P_{ij}\big|_{\mathfrak{S}_\nu^\circ}\right] \cdot \mathbf{f}(\mathbf{x})\right)_{i,j=1,2,\ldots,n}\right\|_{\max} \\
&= n \cdot \max_{i,j=1,2,\ldots,n} \left|\left(\left[\nabla M_{ij}(\mathbf{x}) - \nabla P_{ij}\big|_{\mathfrak{S}_\nu^\circ}\right] \cdot \mathbf{f}(\mathbf{x})\right)\right| \\
&\leq n \cdot \max_{i,j=1,2,\ldots,n} \left\|\nabla M_{ij}(\mathbf{x}) - \nabla P_{ij}\big|_{\mathfrak{S}_\nu^\circ}\right\|_1 \|\mathbf{f}(\mathbf{x})\|_\infty \\
&\leq n\, \|\mathbf{f}\|_{C^0(\Omega;\mathbb{R}^n)} \max_{i,j=1,2,\ldots,n} \left\|\nabla M_{ij}(\mathbf{x}) - \nabla (M_c)_{ij}\big|_{\mathfrak{S}_\nu^\circ} + \nabla (M_c)_{ij}\big|_{\mathfrak{S}_\nu^\circ} - \nabla P_{ij}\big|_{\mathfrak{S}_\nu^\circ}\right\|_1 \\
&\leq n\, \|\mathbf{f}\|_{C^0(\Omega;\mathbb{R}^n)} \max_{i,j=1,2,\ldots,n} \left(h_{\text{tri.}}\gamma B_M + \left\|\nabla (M_c)_{ij}\big|_{\mathfrak{S}_\nu^\circ} - \nabla P_{ij}\big|_{\mathfrak{S}_\nu^\circ}\right\|_1\right) \\
&\leq n\, B_{\mathbf{f}} \left(h_{\text{tri.}}\gamma B_M + \max_{i,j=1,2,\ldots,n} \left\|\nabla (M_c)_{ij}\big|_{\mathfrak{S}_\nu^\circ} - \nabla P_{ij}\big|_{\mathfrak{S}_\nu^\circ}\right\|_1\right).
\end{aligned}$$

We now consider the last term; we have for all $i, j = 1, 2, \ldots, n$ by (2.4), norm equivalence relations, and (2.1), that

$$\begin{aligned}
\left\|\nabla (M_c)_{ij}\big|_{\mathfrak{S}_\nu^\circ} - \nabla P_{ij}\big|_{\mathfrak{S}_\nu^\circ}\right\|_1 &\leq n \left\|X_\nu^{-1} \begin{pmatrix} (M_c)_{ij}(\mathbf{x}_1) - (M_c)_{ij}(\mathbf{x}_0) - P_{ij}(\mathbf{x}_1) + P_{ij}(\mathbf{x}_0) \\ \vdots \\ (M_c)_{ij}(\mathbf{x}_n) - (M_c)_{ij}(\mathbf{x}_0) - P_{ij}(\mathbf{x}_n) + P_{ij}(\mathbf{x}_0) \end{pmatrix}\right\|_\infty \\
&\leq n \left\|X_\nu^{-1}\right\|_\infty \left\|\begin{pmatrix} (M_c)_{ij}(\mathbf{x}_1) - P_{ij}(\mathbf{x}_1) \\ \vdots \\ (M_c)_{ij}(\mathbf{x}_n) - P_{ij}(\mathbf{x}_n) \end{pmatrix} + \begin{pmatrix} -(M_c)_{ij}(\mathbf{x}_0) + P_{ij}(\mathbf{x}_0) \\ \vdots \\ -(M_c)_{ij}(\mathbf{x}_0) + P_{ij}(\mathbf{x}_0) \end{pmatrix}\right\|_\infty \\
&\leq 2n \left\|X_\nu^{-1}\right\|_\infty \max_{\substack{i,j=1,2,\ldots,n \\ k=0,1,\ldots,n}} \left|(M_c)_{ij}(\mathbf{x}_k) - P_{ij}(\mathbf{x}_k)\right| \\
&\leq 2n \left\|X_\nu^{-1}\right\|_\infty \max_{\substack{i,j=1,2,\ldots,n \\ k=0,1,\ldots,n}} \left|M_{ij}(\mathbf{x}_k) - \widetilde{M}_{ij}(\mathbf{x}_k)\right| \\
&\leq 2n^2 \left\|X_\nu^{-1}\right\|_1 \varepsilon^* \leq 2n^2 \frac{\ell b}{h_{\text{tri.}}} \varepsilon^*.
\end{aligned} \tag{2.9}$$

Putting the pieces together, we obtain

$$\begin{aligned}\|F(M)(\mathbf{x})-A_\nu(\mathbf{x})\|_2 &\le 2\,\|M(\mathbf{x})-P(\mathbf{x})\|_2\,\|D\mathbf{f}(\mathbf{x})\|_2+\left\|M'(\mathbf{x})-\left(\nabla P_{ij}^{\nu}\cdot\mathbf{f}(\mathbf{x})\right)_{i,j=1,2,\dots,n}\right\|_2\\ &\le B_{\mathbf{f}}\left[2nh_{\text{tri.}}^2B_M+2\varepsilon^*+nh_{\text{tri.}}\gamma B_M+2n^3\frac{\ell b}{h_{\text{tri.}}}\varepsilon^*\right]\\ &\le B_{\mathbf{f}}\left[(\gamma+2h_{\text{tri.}})nh_{\text{tri.}}B_M+2\left(1+\frac{n^3\ell b}{h_{\text{tri.}}}\right)\varepsilon^*\right].\end{aligned}$$

As $\mathbf{x}\in\mathcal{D}_{\mathcal{T}}$ was arbitrary as well as $\mathfrak{S}_\nu\in\mathcal{T}$ such that $\mathbf{x}\in\mathfrak{S}_\nu$, we have shown (2.6). □

2.10 Remark The above estimates show the difficulty of dealing with several numerical approximations and provide an insight for the way the algorithm should work to succeed in providing a good estimate. To be more precise (see also the statement of Theorem 2.11): First we fix the triangulation with $h_{\text{tri.}}\le h_{\text{tri.}}^*$ sufficiently small. Then we choose the parameters $H\ge H^*$ and $N\ge N^*$ sufficiently large to obtain such a small $\varepsilon^*>0$, that even $\varepsilon^*/h_{\text{tri.}}$ is small. Thus, we can make the total error term smaller than any given $\varepsilon>0$.

The final statement of this section is a converse proposition to Theorem 2.7; starting from a numerical approximation of the contraction metric on a fine grid, the CPA interpolation of these values delivers a function that satisfies the conditions of the Verification Problem 2.6. Note that we need to assume at least $\mathbf{f}\in C^5(\mathbb{R}^n;\mathbb{R}^n)$ to use Theorem 1.8 and the assumptions on Ω are required to use Theorem 2.9. The scaled standard triangulation satisfies the assumptions for $b\ge 2\sqrt{n}$ and a sufficiently small scaling parameter.

2.11 Theorem (Computed and verified CPA contraction metric)
Assume that $\mathbf{x}_0$ is an exponentially stable equilibrium of the system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$, where $\mathbf{f}\in C^{2(p+1)+1}(\mathbb{R}^n;\mathbb{R}^n)$ with $p\in\mathbb{N}_+$. Let $\Omega\subset\overline{\Omega}\subset\mathcal{A}(\mathbf{x}_0)$ be a bounded open set with C^1 boundary and $K\subset\Omega$ be a positively invariant and compact set such that $\mathbf{x}_0\in K^\circ$. Let $M\in C^{2(p+1)+1}(\mathcal{A}(\mathbf{x}_0);\mathbb{S}^{n\times n})$ be the solution of the PDE (1.4) from Theorem 1.7 for some $C\in C^{2(p+1)}(\mathcal{A}(\mathbf{x}_0);\mathbb{S}^{n\times n})$, $C(\mathbf{x})\succ 0$ for all $\mathbf{x}\in\mathcal{A}(\mathbf{x}_0)$.

Fix a compact set $\widetilde{K}\subset K^\circ$ and constants

$$\ell\ge 1,\quad b>0,\quad B_2^*\ge\max_{\substack{\mathbf{x}\in K\\ i,j,k=1,2,\dots,n}}\left|\frac{\partial^2 f_k}{\partial x_i\partial x_j}(\mathbf{x})\right|,\quad\text{and}\quad B_3^*\ge\max_{\substack{\mathbf{x}\in K\\ i,j,k,l=1,\dots,n}}\left|\frac{\partial^3 f_l}{\partial x_i\partial x_j\partial x_k}(\mathbf{x})\right|.$$

Then there exists a constant $h_{\text{tri.}}^*>0$ such that for any $(h_{\text{tri.}},b,\ell)$-bounded triangulation $\mathcal{T}$ with $\widetilde{K}\subset D_{\mathcal{T}}^\circ\subset D_{\mathcal{T}}\subset K^\circ$ and $0<h_{\text{tri.}}\le h_{\text{tri.}}^*$, there exists a time interval length $H^*>0$, such that for all fixed and finite $H\ge H^*$, there exists a $N^*=2^pq^*$, $q^*\in\mathbb{N}_+$, such that for all $N=2^pq$, where $q\in\mathbb{N}_+$ and $q\ge q^*$, the following holds:

Suppose that $\widetilde{M}$ is the numerical approximation of M as described in (1.8) using the RK4 and AB4 methods and the Romberg-like numerical integration on the time interval $[0,H]$; all with step size $h=H/N$. Fix the constants of Verification Problem 2.6 as follows for all $\mathfrak{S}_\nu\in\mathcal{T}$, $\mathbf{x}_k\in\mathcal{V}_{\mathcal{T}}$, and $1\le i,j\le n$ (see Remark 2.12 for the constant ε_0):

$$P_{ij}(\mathbf{x}_k)=\widetilde{M}_{ij}(\mathbf{x}_k),\quad C_\nu=\max\{\|P(\mathbf{x})\|_2\,:\,\mathbf{x}\text{ vertex of }\mathfrak{S}_\nu\},\quad D_\nu=\max_{i,j=1,\dots,n}\left\|\nabla P_{ij}|_{\mathfrak{S}_\nu^\circ}\right\|_1,$$

$$B_2^*\ge B_{2,\nu}\ge\max_{\substack{\mathbf{x}\in\mathfrak{S}_\nu\\ i,j,k=1,2,\dots,n}}\left|\frac{\partial^2 f_k}{\partial x_i\partial x_j}(\mathbf{x})\right|,\quad B_3^*\ge B_{3,\nu}\ge\max_{\substack{\mathbf{x}\in\mathfrak{S}_\nu\\ i,j,k,l=1,\dots,n}}\left|\frac{\partial^3 f_l}{\partial x_i\partial x_j\partial x_k}(\mathbf{x})\right|.$$

Then the constraints of Verification Problem 2.6 are fulfilled by these values. In particular, we can assert that the CPA interpolation P of numerical approximation $\widetilde{M}$ to the contraction metric M on $\mathcal{T}$ is a contraction metric on $\widetilde{K}$.

2.12 Remark A formula for the constant $\varepsilon_0 > 0$ of Verification Problem 2.6 is given in the proof below. Note that in the verification process one can simply check $0 \prec P(\mathbf{x}_k)$ in (VP1) for all vertices $\mathbf{x}_k$. If it is fulfilled then the existence of an $\varepsilon_0 > 0$ such that $\varepsilon_0 I \preceq P(\mathbf{x}_k)$ for all vertices follows. If one wants to actually solve the semidefinite optimization problem, however, then one must fix $\varepsilon_0 > 0$ in advance, but its numerical value is not of importance for the feasibility of the problem. The reason for this is that if P is a contraction metric, then so is cP for any constant $c > 0$. Hence, if there is a feasible solution for the value for ε_0 given in formula (2.12), there is also a feasible solution when using the value $\varepsilon_0' > 0$ for ε_0, just multiply all the variables $P(\mathbf{x}_k)$, C_ν, and D_ν by $\varepsilon_0'/\varepsilon_0$.

Proof: First note that since $\overline{\Omega}$ is compact and M is positive definite by Theorem 1.7, there are constants $\lambda_0, \lambda_1, \Lambda_0 > 0$ such that for all $\mathbf{x} \in \overline{\Omega}$ we have

$$\begin{aligned} \lambda_0 I &\preceq M(\mathbf{x}) \preceq \Lambda_0 I, & (2.10)\\ \lambda_1 I &\preceq C(\mathbf{x}). & (2.11)\end{aligned}$$

Furthermore, define

$$\begin{aligned} \varepsilon_0 &:= \min\left\{\frac{\lambda_0}{5}, \frac{\lambda_1}{6}\right\}, & (2.12)\\ C^* &:= \Lambda_0 + \frac{3}{5}\lambda_0, & (2.13)\\ \gamma &:= 1 + \frac{bn^{\frac{3}{2}}}{2}, \end{aligned}$$

and fix, with $B_M := \|M\|_{C^2(\Omega;\mathbb{S}^{n\times n})}$ and $B_{\mathbf{f}} := \|\mathbf{f}\|_{C^1(\Omega;\mathbb{R}^n)}$, the constant $0 < h^*_{\text{tri.}} \leq 1$ such that

$$(h^*_{\text{tri.}})^2 \leq \frac{\lambda_0}{5} \cdot \frac{1}{nB_M}, \quad (h^*_{\text{tri.}})^2 \leq \frac{\lambda_1}{6} \cdot \frac{1}{n^2(1+4\sqrt{n})(1+\gamma)B_2^* B_M + 2n^3 B_3^* C^*} \tag{2.14}$$
$$\text{and} \quad h^*_{\text{tri.}} \leq \frac{\lambda_1}{6} \cdot \frac{1}{n(\gamma+2)B_M B_{\mathbf{f}}}.$$

Choose an arbitrary $h_{\text{tri.}}$ fulfilling $0 < h_{\text{tri.}} \leq h^*_{\text{tri.}}$ and define

$$\varepsilon^* := \min\left\{\frac{3\lambda_0}{5}, \frac{\lambda_1}{6} \cdot \min\left\{\frac{1}{2n^4\ell b(1+4\sqrt{n})B_2^* h_{\text{tri.}}}, \frac{1}{2B_{\mathbf{f}}}, \frac{h_{\text{tri.}}}{2\ell b n^3 B_{\mathbf{f}}}\right\}\right\}. \tag{2.15}$$

Now, for $\varepsilon^* > 0$ given by (2.15) let $H^* > 0$ be a constant as in Theorem 1.8. Then, for every $H \geq H^*$ there exists a constant $N^* = 2^p q^*$, $q^* \in \mathbb{N}_+$, such that for every $N = 2^p q$, $q \in \mathbb{N}_+$ and $q \geq q^*$, the estimates (2.5) and (2.6) hold true for the CPA interpolation P of the numerically computed approximation $\widetilde{M}$ to the contraction metric M. Further note that, by definition, the matrices $P(\mathbf{x}_k) := \widetilde{M}(\mathbf{x}_k)$ are symmetric for all vertices $\mathbf{x}_k \in \mathcal{V}_\mathcal{T}$.

We first assert that the constraints (VP1) in Verification Problem 2.6 are fulfilled.

We have for all $\mathbf{x} \in D_{\mathcal{T}} \subset K \subset \Omega$ that

$$
\begin{aligned}
P(\mathbf{x}) &= M(\mathbf{x}) - (M(\mathbf{x}) - P(\mathbf{x})) \succeq M(\mathbf{x}) - \|M(\mathbf{x}) - P(\mathbf{x})\|_2 I \\
&\succeq \left(\lambda_0 - [\varepsilon^* + n\, h_{\text{tri.}}^2 B_M]\right) I \succeq \left(\lambda_0 - \left[\frac{3\lambda_0}{5} + n\frac{\lambda_0}{5nB_M}\, B_M\right]\right) I \succeq \frac{\lambda_0}{5} I \\
&\succeq \varepsilon_0 I,
\end{aligned}
$$

where we used (2.5), (2.10), (2.15), (2.14), and (2.12); recall that for $A \in \mathbb{S}^{n\times n}$ we have $-\|A\|_2 I \preceq A \preceq \|A\|_2 I$ and $\|A\|_2 \geq 0$ is the smallest number with this property. Thus, the constraints (VP1) hold true.

We now proceed to show that the constraints (VP2) in Verification Problem 2.6 are fulfilled. First, we show that we have $C_\nu \leq C^*$ for all ν. For all $\mathbf{x} \in D_{\mathcal{T}}$, similarly to above, we get that

$$
P(\mathbf{x}) = M(\mathbf{x}) - (M(\mathbf{x}) - P(\mathbf{x})) \preceq (\Lambda_0 + \varepsilon^*)\, I \preceq \left(\Lambda_0 + \frac{3\lambda_0}{5}\right) I = C^* I,
$$

which shows $C_\nu \leq C^*$ for all ν by the definition of C_ν.

In order to obtain an upper bound on the D_νs consider a simplex $\mathfrak{S}_\nu \in \mathcal{T}$ and let $1 \leq i, j \leq n$. Then, denoting the CPA interpolation of M on the triangulation $\mathcal{T}$ by M_c we have

$$
\begin{aligned}
\left\|\nabla P_{ij}\big|_{\mathfrak{S}_\nu^\circ}\right\|_1 &\leq \left\|\nabla P_{ij}\big|_{\mathfrak{S}_\nu^\circ} - \nabla (M_c)_{ij}\big|_{\mathfrak{S}_\nu^\circ}\right\|_1 + \left\|\nabla (M_c)_{ij}\big|_{\mathfrak{S}_\nu^\circ}\right\|_1 \\
&\leq 2n^2 \frac{\ell b}{h_{\text{tri.}}}\varepsilon^* + (1 + h_{\text{tri.}}\gamma)\, \|M\|_{C^2(\Omega;\mathbb{S}^{n\times n})} \\
&= \frac{2n^2 \ell b}{h_{\text{tri.}}}\varepsilon^* + (1+\gamma) B_M, \qquad (2.16)
\end{aligned}
$$

where we used inequality (2.9), $h_{\text{tri.}} \leq 1$, and [27, Lemma 4.5]. Thus, we have that D_ν is less than or equal to the right-hand side of inequality (2.16) for all ν.

We now derive an upper bound on the error term $h_\nu^2 E_\nu$. Note that, using the formula (2.3) for E_ν, (2.16), (2.14), and (2.15), we have,

$$
\begin{aligned}
h_\nu^2 E_\nu &= h_{\text{tri.}}^2 \left(n^2(1 + 4\sqrt{n}) B_{2,\nu} D_\nu + 2\, n^3 B_{3,\nu} C_\nu\right) \qquad (2.17) \\
&\leq n^2 (1 + 4\sqrt{n}) B_2^* D_\nu h_{\text{tri.}}^2 + 2\, n^3 B_3^* C^* h_{\text{tri.}}^2 \\
&\leq 2n^4 \ell b (1 + 4\sqrt{n}) B_2^* h_{\text{tri.}} \cdot \varepsilon^* + \left[n^2(1 + 4\sqrt{n})(1+\gamma) B_2^* B_M + 2\, n^3 B_3^* C^*\right] \cdot (h_{\text{tri.}}^*)^2 \\
&\leq \frac{\lambda_1}{6} + \frac{\lambda_1}{6} = \frac{\lambda_1}{3}.
\end{aligned}
$$

To conclude the proof fix a simplex $\mathfrak{S}_\nu \in \mathcal{T}$ and let $\mathbf{x}_k$ be one of its vertices. Then $\mathbf{x}_k \in D_{\mathcal{T}} \subset K \subset \Omega$ and we have by (2.6), (2.14), and (2.15), that

$$
\begin{aligned}
\|F(M)(\mathbf{x}_k) - A_\nu(\mathbf{x}_k)\|_2 &\leq \left[(\gamma + 2h_{\text{tri.}}) n h_{\text{tri.}} B_M + 2\left(\frac{\ell b n^3}{h_{\text{tri.}}} + 1\right)\varepsilon^*\right] B_{\mathbf{f}} \qquad (2.18) \\
&\leq n(\gamma+2) B_M B_{\mathbf{f}} \cdot h_{\text{tri.}} + 2\ell b n^3 B_{\mathbf{f}} \cdot \frac{\varepsilon^*}{h_{\text{tri.}}} + 2B_{\mathbf{f}} \cdot \varepsilon^* \\
&\leq \frac{\lambda_1}{6} + \frac{\lambda_1}{6} + \frac{\lambda_1}{6} = \frac{\lambda_1}{2} \qquad (2.19)
\end{aligned}
$$

and thus

$$A_\nu(\mathbf{x}_k) \preceq \|A_\nu(\mathbf{x}_k) - F(M)(\mathbf{x}_k)\|_2 I + F(M)(\mathbf{x}_k) \preceq \frac{\lambda_1}{2} I - C(\mathbf{x}_k) \preceq \frac{\lambda_1}{2} I - \lambda_1 I = -\frac{\lambda_1}{2} I.$$

Combining this last estimate with (2.17) delivers

$$A_\nu(\mathbf{x}_k) + h_\nu^2 E_\nu I \preceq -\frac{\lambda_1}{2} I + \frac{\lambda_1}{3} I = -\frac{\lambda_1}{6} I \preceq -\varepsilon_0 I$$

and we have shown that the constraints (VP2) are fulfilled for an arbitrary simplex $\mathfrak{S}_\nu \in \mathcal{T}$ and an arbitrary vertex $\mathbf{x}_k$ of $\mathfrak{S}_\nu$. This concludes the proof. □

3 Examples

In this section we apply our method to two examples from the literature. In both examples we compute a contraction metric on a compact positively invariant set $\mathcal{F}$ and thus establish the existence of a unique equilibrium in $\mathcal{F}$ that is exponentially stable. Further, $\mathcal{F}$ is a subset of its basin of attraction.

In order to compute a positively invariant set $\mathcal{F}$ for a dynamical systems given by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ we use the method described in [28] and motivated by [23]. In this method one first solves numerically the Zubov-like PDE

$$\sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(\mathbf{x}) f_i(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = -\sqrt{\delta^2 + \|\mathbf{f}(\mathbf{x})\|_2^2}, \tag{3.1}$$

with $\delta = 10^{-8}$, using collocation with radial basis functions. Then we interpolate the numerical solution V to (3.1) by the CPA interpolation V_P on a simplicial complex with simplices of diameter h_ν and verify for which simplices the condition $\nabla V_P^\nu \cdot \mathbf{f}(\mathbf{x}_k) + h_\nu^2 E_\nu^* < 0$ holds true for all vertices $\mathbf{x}_k$, similar to the verification problem for contraction metrics, which shows that the (Dini-)orbital derivative satisfies $(V_P)'_+(\mathbf{x}) < 0$ on the simplex. Here E_ν^* is an error constant tailor made for this problem, for the details see [23]. In this area the function V_P is decreasing along solution trajectories and a sublevel set $\mathcal{F} := \{\mathbf{x} \in \mathbb{R}^n : V_P(\mathbf{x}) \le c\}$ is necessarily positively invariant, if its boundary is fully contained in this area. Hence, we only require $\nabla V_P(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) < 0$ on the level set $\{\mathbf{x} \in \mathbb{R}^n : V_P(\mathbf{x}) = c\}$, not on the whole sublevel set. We refer to V_P as *Lyapunov-like* function.
The *failing points* of the Lyapunov-like function (see for example Figure 2) are the points where the function V_P does not satisfy the decrease condition mentioned above. As stated above, in order to obtain a positively invariant set, we need to find a sublevel set of the function such that its boundary does contain any of these failing points.

3.1 Example (Hourglass Sink) *We consider the following system discussed in [18, Ex. 6.4], [32, Ex. 4.2], and [29, Ex. 2.1]*

$$\begin{cases} \dot{x} &=& x(x^2+y^2-1) - y(z^2+1) \\ \dot{y} &=& y(x^2+y^2-1) + x(z^2+1) \\ \dot{z} &=& 10z(z^2-1) \end{cases} \tag{3.2}$$

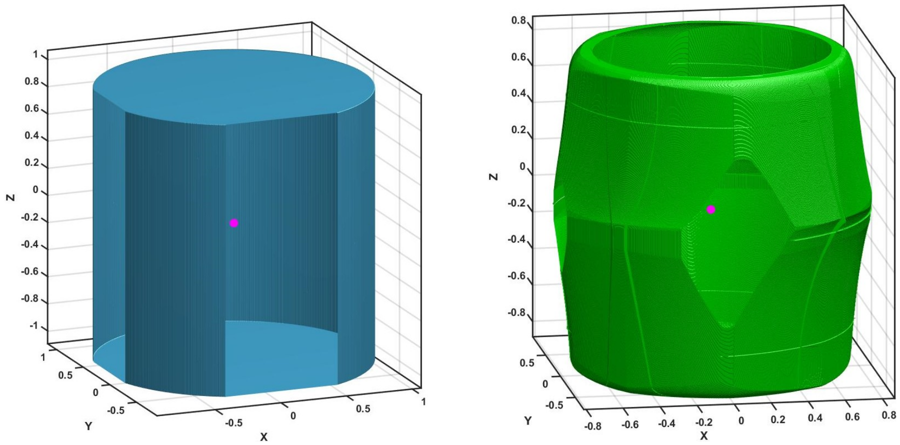

Figure 1: Example 3.1: the blue surface (left) is the boundary between the area where the verification condition (VP1) is satisfied and where it is not satisfied, while the green surface (right) describes the verification condition (VP2) suggested by the Numerical Integration-CPA method. Hence, P is a contraction metric within the intersection of the areas bounded by both surfaces.

For this example we can analytically determine the basin of attraction of the asymptotically stable equilibrium $(0,0,0)$*:*

$$\mathcal{A}(0,0,0) = \left\{(x,y,z) \in \mathbb{R}^3 \,:\, x^2 + y^2 < 1, |z| < 1\right\}$$

and thus can compare the subset obtained by our method to the actual basin of attraction. We fixed the time horizon $H = 10$ *and the number of time steps as* $N = 1,000$*. Then, we used a uniform triangulation, cf. [36], of the cube* $[-1.25,\, 1.25]^3$ *with* $1,001^3$ *vertices. In Figure 1, the left plot shows the boundary of the area where (VP1) is not satisfied, which is outside* $[-0.7, 0.7] \times [-0.7,\, 0.7] \times [-0.85,\, 0.85]$*, and the right plot displays the boundary of the area where the approximation fails to satisfy (VP2), which is outside* $[-0.5,\, 0.5]^2 \times [-0.85,\, 0.85]$*.*

Finally, we have determined a compact, connected, and positively invariant set within the domain of our computed contraction metric. This is done by computing level sets of a Lyapunov-like function V *using a similar approach to [23]. We computed a numerical solution to* $\nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = -\sqrt{\delta^2 + \|\mathbf{f}(\mathbf{x})\|_2^2}$*,* $\mathbf{x} \in \mathbb{R}^3$*, using the RBF method, with* $\mathbf{f}$ *being the right-hand side of (3.2) and* $\delta^2 = 10^{-8}$*. We used* $N = 1,331$ *collocation points given by* $X = 0.13 \cdot \mathbb{Z}^3 \cap [-0.65, 0.65]^3$ *for the RBF method, set* $c = 0.9$*, and used the Wendland function* $\psi_{5,3}(r) = (1-r)_+^8(32r^3 + 25r^2 + 8r + 1)$*; here* $(1-r)_+^8 := (\max\{0, 1-r\})^8$*. For the CPA verification we used a uniform triangulation of the cube* $[-0.75, 0.75]^3$ *with* 801^3 *vertices. In Figure 2, the yellow dots indicate the failing points of the Lyapunov-like function. The green surface is the boundary of a set, such that inside it Verification conditions (VP1) and (VP2) are satisfied. The red closed shape is a level set of the Lyapunov-like function and is therefore positively invariant for the dynamics. From the results we can conclude that there is exactly one equilibrium in the set bounded by the red surface, that it is exponentially stable, and that the red set is a subset of its basin of attraction.*

3.2 Example (Balsam fir tree, Moose, and Wolf) *Let us consider the following system*

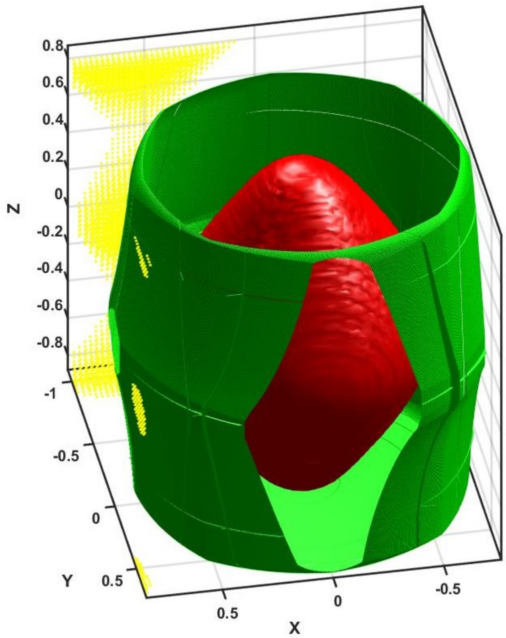

Figure 2: Example 3.1: The yellow dots indicate the failing points of the Lyapunov-like function while the red closed shape is a level set of it. The green surface is the boundary of the area where verification conditions (VP2) are satisfied; the area where the constraints (VP1) are satisfied is a superset of this set, hence both (VP1) and (VP2) are satisfied within the area bounded by the green surface. From the results we can conclude that there is exactly one equilibrium in the set bounded by the red surface, that it is exponentially stable, and that the red set is a subset of its basin of attraction.

from [50], also discussed in [1, Ex. 7.10],

$$\begin{cases} \dot{x} &=& x(1-x) - xy \\ \dot{y} &=& y(1-y) + xy - yz \\ \dot{z} &=& z(1-z) + yz \end{cases} \tag{3.3}$$

in which we denote by $x(t), y(t), z(t)$, respectively, the populations of trees, moose, and wolves at time t. The fir trees are eaten by the moose, moose are eaten by wolves, and the change in the wolf population affects the trees.
The model assumes that these populations in isolation are subject to logistic growth and the effect of interaction between species is proportional to the product of the populations. For simplicity, we assume that all the parameters, namely, the growth rate, the carrying capacity, and the effect of iteration, on which the behavior of solutions depends, are 1.
The equilibria of the system are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,0,1)$, and $(2/3, 1/3, 4/3)$. The last one has all populations present, and turns out to be exponentially stable. We are interested in its basin of attraction.

We fixed the time horizon $H = 50$, and the number of step-sizes $N = 1,000$. Then, we used a uniform triangulation of the cube $[-0.05,\ 1.1] \times [-0.05,\ 1.1] \times [0.15,\ 2.5]$ with 301^3 vertices. In Figure 3, from left to right, the first plot shows the boundary of the area where (VP1) is not satisfied in blue and the equilibria of the system are represented as magenta points.

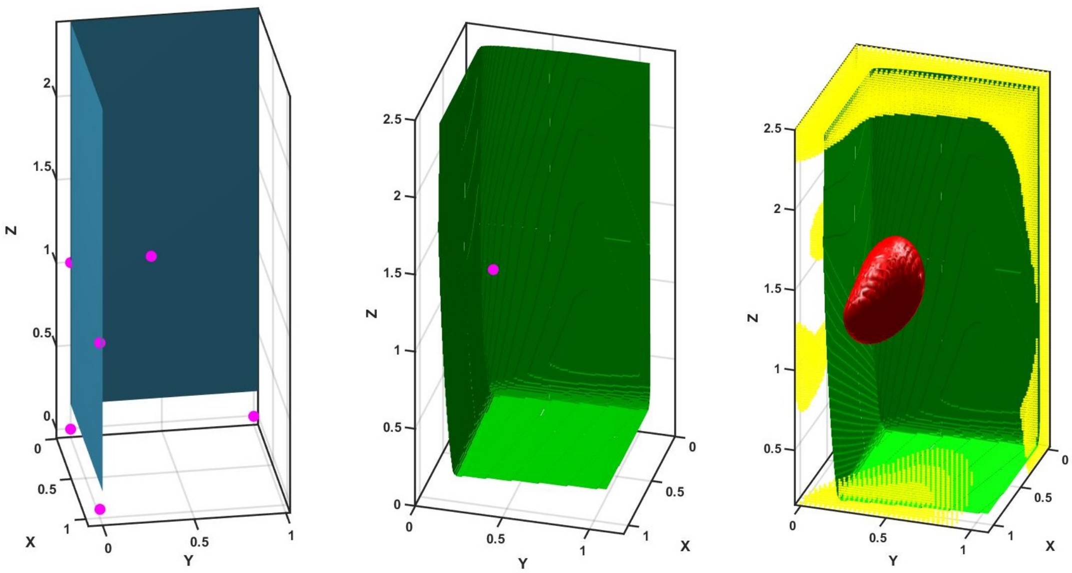

Figure 3: Example 3.2. Left: The magenta dots are the equilibria. The blue surface is the boundary between the area where the verification condition (VP1) is satisfied and where it is not satisfied. Middle: The green surface is the boundary between the area where the verification condition (VP2) is satisfied and where it is not satisfied; hence, P is a contraction metric within the intersection of the areas bounded by the blue and green sets. Clearly, in this example the blue set is far behind the green set and does not have any effect on limiting the area. Right: This plot shows a level set of a Lyapunov-like function in red and the failing points of that function in yellow. The set bounded by the red surface is positively invariant subset of the basin of attraction of an exponentially stable equilibrium in its interior.

The middle plot displays the boundary of the area where the approximation fails to satisfy (VP2) in green. The right plot shows a compact, connected and positively invariant set, which is bounded by the red level set of V_P; the yellow points indicate where the Lyapunov-like function fails to satisfy $(V_P)'_+(\mathbf{x}) < 0$. The level set bounds the sub-level set of the Lyapunov-like function within the domain of our computed contraction metric, which is thus a subset of the basin of attraction of the equilibrium $(2/3, 1/3, 4/3)$. The Lyapunov-like function was computed as in the last example and with the same parameters, using $N = 576$ collocation points given by

$$X = \left(0.12 \cdot \mathbb{Z}^3\right) \cap \left\{[0.15, 0.85]^2 \times [0.15, 2.15]\right\}.$$

3.3 Robustness of the metric

An advantage of contraction metrics in comparison to Lyapunov functions is that they are robust to perturbations of the dynamics. A contraction metric remains a contraction metric for sufficiently small perturbations, even if the equilibrium is shifted. To demonstrate this robustness we consider perturbations of the system (3.3) from Example 3.2, namely

$$\left\{\begin{array}{rcl} \dot{x} & = & x(1-x) - x(y-\varepsilon) \\ \dot{y} & = & y(1-y) + (x-\varepsilon)y - y(z-\varepsilon) \\ \dot{z} & = & z(1-z) + (y-\varepsilon)z. \end{array}\right. \tag{3.4}$$

In particular, we consider a small perturbation with $\varepsilon = 0.03$, which changes the position of the stable equilibrium to $(0.68, 0.35, 1.32)$, and a large one with $\varepsilon = 0.1$, which changes the position of the stable equilibrium to $(0.7, 0.4, 1.3)$. We use the contraction metric and Lyapunov-like function that were computed for the unperturbed system (3.2) and check, whether and where they are still valid for the perturbed system. In particular, we check the verification condition (VP2), where $\mathbf{f}$ is replaced by the right-hand side of the perturbed system (3.4) (and similarly for its derivatives); note that condition (VP1), the positive definiteness of P, trivially holds as it is the same metric as for the unperturbed system.

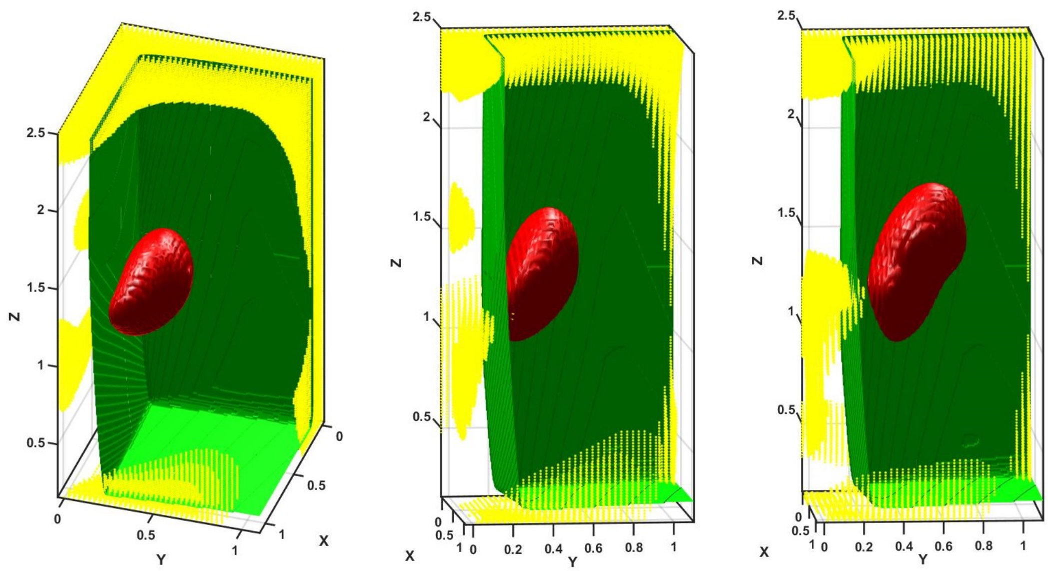

Figure 4: System (3.4): The green surface indicates the boundary of the area where (VP2) is satisfied. The yellow points indicate where the Lyapunov-like function fails to satisfy $(V_P)'_+(\mathbf{x}) < 0$, and the red set is a level set of V_P. From left to right, the first plot is for the small perturbation $\varepsilon = 0.03$, the second plot shows the problem with the large perturbation $\varepsilon = 0.1$, and the last plot displays a new Lyapunov-like function computed for the large perturbation $\varepsilon = 0.1$. Hence, the set bounded by the red surface is a positively invariant subset of the basin of attraction of an exponentially stable equilibrium in the interior of the set for the perturbed system (3.4) with $\varepsilon = 0.03$ (left) and $\varepsilon = 0.1$ (right).

Figure 4, left, shows that for the small perturbation $\varepsilon = 0.03$, both the contraction metric and the Lyapunov-like function still remain valid in a large area. For the large perturbation $\varepsilon = 0.1$ the Lyapunov-like function fails to satisfy $(V_P)'_+(\mathbf{x}) < 0$ in many more points and we are not able to find a suitable sub-level set, while the contraction metric remains valid. A bigger level set cuts the green surface and a smaller one leaves out some of the failing points around the equilibrium. Hence, we have kept the contraction metric for the unperturbed system, but have computed a new Lyapunov-like function for the perturbed system; the results are shown in Figure 4, right.

4 Conclusion

In this paper we introduced a method to construct and verify a contraction metric for a dynamical system with an exponentially stable equilibrium. A contraction metric is a tool to show the stability of an equilibrium and to determine a subset of its basin of attraction. Its advantage is that it is robust with respect to perturbations of the dynamical system, including perturbing the position of the equilibrium.

Our method consists of two sub-routines, first numerically solving an integral to find the values of a contraction metric at several points and then interpolating these values to obtain a continuous and affine function (CPA) over each simplex of a fixed triangulation. The properties for a contraction metric are rigorously verified, proving that the computed metric is in fact a contraction metric. We have provided detailed error estimates and we can assert that the computed metric is truly a contraction metric and not an approximation. Moreover, we have proved a constructive converse theorem, i.e. we have shown that the construction and verification always succeeds, if the step sizes in the numerical approximation and the simplices in the triangulation are sufficiently small.

When compared to other methods, like the RBF-CPA contraction metrics in [27, 28], the new method proposed in this paper is more easily parallelizable, because the computations of the values of the metric at the vertices of the triangulation are independent. Compared to Lyapunov functions, the computation of a contraction metric is computationally more demanding as we construct a matrix-valued function. However, it is robust with respect to perturbations of the system.

References

[1] R. Agarwal, S. Hodis, and D. O'Regan, *500 examples and problems of applied differential equations*, Springer, 2019.

[2] E. Akin, *The general topology of dynamical systems*, American Mathematical Society, 2010.

[3] S. Albertsson, P. Giesl, S. Gudmundsson, and S. Hafstein, *Simplicial complex with approximate rotational symmetry: A general class of simplicial complexes*, J. Comp. Appl. Math. **363** (2020), 413–425.

[4] Z. Aminzare and E. Sontag, *Contraction methods for nonlinear systems: A brief introduction and some open problems*, Proceedings of the 53rd IEEE Conference on Decision and Control, 2014, pp. 3835–3847.

[5] J. Auslander, *Generalized recurrence in dynamical systems*, Contr. to Diff. Equ. **3** (1964), 65–74.

[6] E.M. Aylward, P. A. Parrilo, and J.-J. Slotine, *Stability and robustness analysis of nonlinear systems via contraction metrics and SOS programming*, Automatica **44** (2008), 2163–2170.

[7] P. Bernhard and S. Suhr, *Lyapounov functions of closed cone fields: From Conley theory to time functions*, Commun. Math. Phys. **359** (2018), 467–498.

[8] N. Bhatia and G. Szegő, *Dynamical systems: Stability theory and applications*, Springer, Berlin. Lecture Notes in Mathematics 35, 1967.

[9] J. Björnsson, P. Giesl, S. Hafstein, C. Kellett, and H. Li, *Computation of continuous and piecewise affine Lyapunov functions by numerical approximations of the Massera construction*, Proceedings of the CDC, 53rd IEEE Conference on Decision and Control (Los Angeles (CA), USA), 2014.

[10] J. Björnsson, P. Giesl, S. Hafstein, C. Kellett, and H. Li, *Computation of Lyapunov functions for systems with multiple attractors*, Discrete Contin. Dyn. Syst. Ser. A **35** (2015), no. 9, 4019–4039.

[11] G. Borg, *A condition for the existence of orbitally stable solutions of dynamical systems*, Kungl. Tekn. Högsk. Handl. 153, 1960.

[12] G. Chesi, *Domain of Attraction: Analysis and Control via SOS Programming*, Springer, 2011.

[13] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series no. 38, American Mathematical Society, 1978.

[14] B. P. Demidovič, *On the dissipativity of a certain non-linear system of differential equations. I*, Vestnik Moskov. Univ. Ser. I Mat. Meh. **1961** (1961), no. 6, 19–27.

[15] A. Doban, *Stability domains computation and stabilization of nonlinear systems: implications for biological systems*, PhD thesis: Eindhoven University of Technology, 2016.

[16] A. Doban and M. Lazar, *Computation of Lyapunov functions for nonlinear differential equations via a Yoshizawa-type construction*, IFAC-PapersOnLine **49** (2016), no. 18, 29 – 34.

[17] F. Forni and R. Sepulchre, *A differential Lyapunov framework for Contraction Analysis*, IEEE Transactions on Automatic Control **59** (2014), no. 3, 614–628.

[18] P. Giesl, *Construction of global Lyapunov functions using radial basis functions*, Lecture Notes in Math. 1904, Springer, 2007.

[19] P. Giesl, *Converse theorems on contraction metrics for an equilibrium*, J. Math. Anal. Appl. (2015), no. 424, 1380–1403.

[20] P. Giesl, *Computation of a contraction metric for a periodic orbit using meshfree collocation*, SIAM J. Appl. Dyn. Syst. **18** (2019), no. 3, 1536–1564.

[21] P. Giesl and S. Hafstein, *Construction of a CPA contraction metric for periodic orbits using semidefinite optimization*, Nonlinear Anal. **86** (2013), 114–134.

[22] P. Giesl and S. Hafstein, *Revised CPA method to compute Lyapunov functions for nonlinear systems*, J. Math. Anal. Appl. **410** (2014), 292–306.

[23] P. Giesl and S. Hafstein, *Computation and verification of Lyapunov functions*, SIAM J. Appl. Dyn. Syst. **14** (2015), no. 4, 1663–1698.

[24] P. Giesl and S. Hafstein, *Review of computational methods for Lyapunov functions*, Discrete Contin. Dyn. Syst. Ser. B **20** (2015), no. 8, 2291–2331.

[25] P. Giesl and S. Hafstein, *Uniformly regular triangulations for parameterizing Lyapunov functions*, Proceedings of the 18th International Conference on Informatics in Control, Automation and Robotics (ICINCO), 2021, pp. 549–557.

[26] P. Giesl, S. Hafstein, and C. Kawan, *Review on contraction analysis and computation of contraction metrics*, J. Comput. Dyn **10** (2023), no. 1, 1–47.

[27] P. Giesl, S. Hafstein, and I. Mehrabinezhad, *Computation and verification of contraction metrics for exponentially stable equilibria*, J. Comput. Appl. Math. **390** (2021), Paper No. 113332.

[28] P. Giesl, S. Hafstein, and I. Mehrabinezhad, *Computation and verification of contraction metrics for periodic orbits*, J. Math. Anal. Appl. **503** (2021), no. 2, Paper No. 125309, 32.

[29] P. Giesl, S. Hafstein, and I. Mehrabinezhad, *Computing contraction metrics for three-dimensional systems*, IFAC PapersOnLine **54** (2021), no. 9, 297–303.

[30] P. Giesl, S. Hafstein, and I. Mehrabinezhad, *Contraction metrics by numerical integration and quadrature: Uniform error estimate*, Proceedings of the 20th International Conference on Informatics in Control, Automation and Robotics (ICINCO), 2023, p. (submitted).

[31] P. Giesl, S. Hafstein, and I. Mehrabinezhad, *Positively invariant sets for ODEs and numerical integration*, Proceedings of the 20th International Conference on Informatics in Control, Automation and Robotics (ICINCO), 2023, p. (submitted).

[32] P. Giesl and H. Wendland, *Construction of a contraction metric by meshless collocation*, Discrete Contin. Dyn. Syst. Ser. B **24** (2019), no. 8, 3843–3863.

[33] S. Hafstein, *Implementation of simplicial complexes for CPA functions in C++11 using the armadillo linear algebra library.*, Proceedings of the 3rd International Conference on Simulation and Modeling Methodologies, Technologies and Applications (SIMULTECH) (Reykjavik, Iceland), 2013, pp. 49–57.

[34] S. Hafstein, C. Kellett, and H. Li, *Computing continuous and piecewise affine Lyapunov functions for nonlinear systems*, Journal of Computational Dynamics **2** (2015), no. 2, 227 – 246.

[35] S. Hafstein and S. Suhr, *Smooth complete Lyapunov functions for ODEs*, J. Math. Anal. Appl. **499** (2021), no. 1, 125003.

[36] S. Hafstein and A. Valfells, *Efficient computation of Lyapunov functions for nonlinear systems by integrating numerical solutions*, Nonlinear Dynamics **97** (2019), no. 3, 1895–1910.

[37] W. Hahn, *Stability of motion*, Springer, Berlin, 1967.

[38] P. Hartman, *On stability in the large for systems of ordinary differential equations*, Canadian J. Math. **13** (1961), 480–492.

[39] P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

[40] M. Hurley, *Chain recurrence, semiflows, and gradients*, J. Dyn. Diff. Equat. **7** (1995), no. 3, 437–456.

[41] M. Hurley, *Lyapunov functions and attractors in arbitrary metric spaces*, Proc. Amer. Math. Soc. **126** (1998), 245–256.

[42] T. Johansen, *Computation of Lyapunov functions for smooth, nonlinear systems using convex optimization*, Automatica **36** (2000), 1617–1626.

[43] P. Julian, J. Guivant, and A. Desages, *A parametrization of piecewise linear Lyapunov functions via linear programming*, Int. J. Control **72** (1999), no. 7-8, 702–715.

[44] N. N. Krasovskiĭ, *Problems of the Theory of Stability of Motion*, Mir, Moskow, 1959., English translation by Stanford University Press, 1963.

[45] D. Lewis, *Differential equations referred to a variable metric*, Amer. J. Math. **73** (1951), 48–58.

[46] D. C. Lewis, *Metric properties of differential equations*, Amer. J. Math. **71** (1949), 294–312 (English).

[47] W. Lohmiller and J.-J. Slotine, *On Contraction Analysis for Non-linear Systems*, Automatica **34** (1998), 683–696.

[48] A. M. Lyapunov, *Problème général de la stabilité du mouvement*, Ann. of math. Stud. **17**, Princeton, 1907, Ann. Fac. Sci. Toulouse **9**, 203–474. Translation of the russian version, published 1893 in Comm. Soc. math. Kharkow. Newly printed: Ann. of math. Stud. **17**, Princeton, 1949.

[49] S. Marinósson, *Lyapunov function construction for ordinary differential equations with linear programming*, Dynamical Systems: An International Journal **17** (2002), 137–150.

[50] B. McLaren and R. Peterson, *Wolves, moose, and tree rings on isle royale*, Science **266** (1994), no. 5190, 1555–1558.

[51] P. Parrilo, *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimiziation*, PhD thesis: California Institute of Technology Pasadena, California, 2000.

[52] M. Peet and A. Papachristodoulou, *A converse sum of squares Lyapunov result with a degree bound*, IEEE Transactions on Automatic Control **57** (2012), no. 9, 2281–2293.

[53] C. Robinson, *Dynamical systems: Stability, symbolic dynamics, and chaos*, 2. ed., Studies in Advanced Mathematics, CRC Press, 1999.

[54] J. Simpson-Porco and F. Bullo, *Contraction theory on Riemannian manifolds*, Systems Control Lett. **65** (2014), 74–80.

[55] A. Vannelli and M. Vidyasagar, *Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems*, Automatica **21** (1985), no. 1, 69–80.

[56] T. Yoshizawa, *Stability theory by Liapunov's second method*, Publications of the Mathematical Society of Japan, No. 9, The Mathematical Society of Japan, Tokyo, 1966.

[57] V. Zubov, *Methods of A. M. Lyapunov and their application*, Translation prepared under the auspices of the United States Atomic Energy Commission; edited by Leo F. Boron, P. Noordhoff Ltd, Groningen, 1964.