Construction of a contraction metric for time-periodic systems using meshless collocation

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***Abstract*— The existence and stability of a periodic orbit for time-periodic systems as well as its basin of attraction can be determined using a contraction metric. In this paper, we will present a numerical construction method based on meshless collocation with radial basis functions. We will first show the existence of a contraction metric satisfying a partial differential equation and then use meshless collocation to approximately solve it, which results in a contraction metric, if the fill distance is sufficiently small.**

***Index Terms*— Approximation methods, Nonlinear dynamical systems, Partial differential equations.**

I. Introduction

Contraction analysis considers the evolution of the distance between two adjacent solutions; if the distance decreases (contracts), then the long-term behaviour of solutions is the same and solutions belong to the basin of attraction of a periodic orbit under appropriate conditions. The advantage compared to other methods, such as Lyapunov functions, is that the position of the periodic orbit is not required.

To obtain a sufficient and necessary condition, the distance needs to be measured with respect to an appropriate Riemannian metric, a so-called contraction metric. The metric can be described by a matrix-valued function $M(t,x)$, defining a point-dependent scalar product $\langle v,w\rangle_{(t,x)} = v^T M(t,x)w$ for $v,w \in \mathbb{R}^n$. The contraction condition can then be expressed by a differential matrix inequality.

Contraction analysis has been used for autonomous systems, discrete-time systems, switched systems, delay equations [22], control systems, stochastic systems [23] and many more, see e.g. [1], [3], [5], [16], [18]–[20], [24]. It is used for trajectory tracking and the control of systems with periodic forcing, e.g. in satellite dynamics [21] and robotics [26], and it can also provide formal optimality, stability and robustness guarantees in learning-based and data-driven automatic control frameworks [25, Part II].

To construct a contraction metric, we need to find a solution to a differential matrix inequality; for an overview of numerical methods for contraction metrics see [10]. For example, the problem can be reformulated using Linear Matrix Inequalities and, relaxing the requirement that the solution needs to be positive definite, sum-of-squared polynomials (SOS) [2] and be solved using semidefinite optimization. The CPA (continuous piecewise affine) method splits the phase space into simplices and constructs the contraction metric as a piecewise affine function on each simplex, using again semi-definite optimization; see [8] for this method in the time-periodic case. Note that semidefinite optimization is computationally expensive. A further approach to compute contraction metrics, which we will use and explain in more detail in Section III, is based on meshless collocation.

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II. Contraction metric for time-periodic systems

Given $f \in C^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$ with $f(t+T,x) = f(t,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, we consider the time-periodic system

$$\dot{x} \;=\; f(t,x), \qquad x \in \mathbb{R}^n. \tag{1}$$

We define $\phi(t;t_0,x_0) = (t_0 + t \bmod T, x(t))$, where $x(t)$ is the solution of (1) with initial condition $x(t_0) = x_0$. Furthermore, we assume that $\phi(t;t_0,x_0)$ exists for all $t \geq 0$, so that ϕ defines a (semi-)dynamical system on the cylinder $S_T \times \mathbb{R}^n$, where S_T denotes the circle of circumference T with the metric $|t-s| = \min(|t-s|, |t-s+T|, |t-s-T|)$.

A periodic orbit of (1) is defined as $\Gamma = \bigcup_{t\geq 0} \phi(t;t_0,p_0)$ where $\phi(T;t_0,p_0) = \phi(0;t_0,p_0)$. The periodic orbit is called *exponentially stable* if it is stable and exponentially attractive, and its basin of attraction $A(\Gamma)$ is then defined as the set

$$\{(t_0,x_0) \in S_T \times \mathbb{R}^n \mid \lim_{t\to\infty} \operatorname{dist}(\phi(t;t_0,x_0),\Gamma) = 0\},$$

where $\operatorname{dist}((t,x),\Gamma) = \min_{(s,y)\in\Gamma} \operatorname{dist}((t,x),(s,y))$ and dist on the right-hand side is the distance in $S_T \times \mathbb{R}^n$, for more details see e.g. [6]. The existence of a periodic orbit and its basin of attraction can be determined by a contraction metric.

A. Sufficiency

We cite [6, Definition 2.3, Theorem 3.1] for the definition and sufficiency of a contraction metric in this context.

Definition 1: A matrix-valued function $M \in C^1(S_T \times \mathbb{R}^n, \mathbb{S}^{n\times n})$, where $\mathbb{S}^{n\times n}$ denotes the set of symmetric $\mathbb{R}^{n\times n}$ matrices, is called Riemannian metric, if $M(t,x)$ is positive definite for each $(t,x) \in S_T \times \mathbb{R}^n$.

Theorem 2: Let $G \subset S_T \times \mathbb{R}^n$ be a connected, compact and positively invariant set, i.e. $\phi(t;t_0,x_0) \in G$ for all $(t_0,x_0) \in$

G and $t \geq 0$. Let M be a Riemannian metric and assume that $L_M(t,x) < 0$ for all $(t,x) \in G$, where

$$L_M(t,x) = \max_{w\in\mathbb{R}^n, w^T M(t,x) w = 1} \frac{1}{2} w^T \Big[D_x f(t,x)^T M(t,x) + M(t,x) D_x f(t,x) + \dot{M}(t,x) \Big] w,$$

$D_x f(t,x)$ is the Jacobian of $f(t,\cdot)$ and $\dot{M}(t,x)$ the matrix with entries $\left(\frac{\partial M_{ij}(t,x)}{\partial t} + \sum_{k=1}^n \frac{\partial M_{ij}(t,x)}{\partial x_k} f_k(t,x) \right)_{i,j=1,\ldots,n}$.

Then there exists one and only one periodic orbit $\Gamma \subset G$ and Γ is exponentially stable, its basin of attraction satisfies $G \subset A(\Gamma)$ and the largest real part $-\nu_0$ of all Floquet exponents of Γ fulfills $-\nu_0 \leq -\nu := \max_{(t,x)\in G} L_M(t,x)$.

Note that the function $L_M(t,x)$ is continuous and is equal to the largest (real) eigenvalue of the symmetric matrix $\frac{1}{2}\Big[D_x f(t,x)^T M(t,x) + M(t,x) D_x f(t,x) + \dot{M}(t,x) \Big]$. A positively invariant set G can, e.g., be computed numerically as a sublevel set of a Lyapunov-like function [15].

B. Necessity

The existence of a contraction metric has been established in [6, Theorem 4.2]. In this paper, we will prove the existence of a contraction metric satisfying the differential equation (2), so that we can construct it by approximately solving this equation.

Theorem 3: Consider (1) with $f \in C^\sigma(S_T \times \mathbb{R}^n, \mathbb{R}^n)$ and $\sigma \geq 2$. Let Γ be an exponentially stable periodic orbit with basin of attraction $A(\Gamma)$. Let $C \in C^{\sigma-1}(A(\Gamma), \mathbb{S}^{n\times n})$ such that $C(t,x)$ is positive definite for all $(t,x) \in A(\Gamma)$.

Then there exists a unique Riemannian metric $M \in C^{\sigma-1}(A(\Gamma), \mathbb{S}^{n\times n})$ such that

$$D_x f(t,x)^T M(t,x) + M(t,x) D_x f(t,x) + \dot{M}(t,x) = -C(t,x) \tag{2}$$

holds for all $(t,x) \in A(\Gamma)$.

Proof: We fix a compact and positively invariant neighborhood $U \subset A(\Gamma)$ of Γ. For a point $\tilde{x}_0 := (t_0, x_0) \in U$ we consider the linear, non-autonomous, time-periodic system

$$\dot{y} = D_x f(\phi(t;\tilde{x}_0)) y$$

and denote the principal fundamental matrix solution at time t with $y(\theta) = I$ by $\Psi(t,\theta;\tilde{x}_0)$, which is a $C^{\sigma-1}$ function with respect to its arguments. There exists $\theta_0 > 0$ such that Ψ is defined for all $t \geq -\theta_0$. Let p be the point on the periodic orbit at time t_0, i.e. $\phi(t;t_0,p)$ is the periodic solution. Then there exist $c_1, \mu > 0$ such that

$$\|\phi(t;t_0,x_0) - \phi(t;t_0,p)\| \leq c_1 e^{-\mu t}$$

by the exponential stability of the periodic orbit and thus

$$\|D_x f(\phi(t;\tilde{x}_0)) - B(t)\| \leq c_2 e^{-\mu t} \tag{3}$$

since $D_x f$ is locally Lipschitz-continuous, where we denote $B(t) = D_x f(\phi(t;t_0,p))$, which is periodic in t with period T. The principal fundamental matrix solution of $\dot{y} = B(t)y$ with $\tilde{\Psi}(0) = I$ can be written as $\tilde{\Psi}(t) = P(t)e^{tL}$ with T-periodic $P \in C^1(\mathbb{R}, \mathbb{C}^{n\times n})$, such that $P(t)$ is invertible, and $L \in \mathbb{C}^{n\times n}$ is Hurwitz since the periodic orbit is exponentially stable, see, e.g. [9, Chapter 3]. Hence, $\dot{P}(t) + P(t)L = B(t)P(t)$.

Denote $\Psi(t) := \Psi(t,0;\tilde{x}_0)$, $Y(t) = P^{-1}(t)\Psi(t)$ and $Z(t) = P^{-1}(t)\tilde{\Psi}(t)$. Using $\dot{\overline{P^{-1}}}(t) = -P^{-1}(t)B(t) + LP^{-1}(t)$ from $\frac{d}{dt}(P^{-1}(t)P(t)) = 0$ and the formula for $\dot{P}(t)$ above, we have

$$\begin{aligned}
\dot{Y}(t) &= \dot{\overline{P^{-1}}}(t)\Psi(t) + P^{-1}(t) D_x f(\phi(t;\tilde{x}_0))\Psi(t) \\
&= -P^{-1}(t)B(t)\Psi(t) + LP^{-1}(t)\Psi(t) \\
&\quad + P^{-1}(t) D_x f(\phi(t;\tilde{x}_0))\Psi(t) \\
&= LY(t) + P^{-1}(t)[D_x f(\phi(t;\tilde{x}_0)) - B(t)]P(t)Y(t)
\end{aligned}$$

and $\dot{Z} = LZ$. Denoting $A(t) = L + P^{-1}(t)[D_x f(\phi(t;\tilde{x}_0)) - B(t)]P(t)$ and $A = L$, we can apply [7, Lemma A.2], which also holds for complex matrices, to conclude that there exist $c_3, \rho > 0$ such that $\|Y(t)\| \leq c_3 e^{-\rho t}$ and thus

$$\|\Psi(t,0;\phi(\theta;\tilde{x}_0))\| \leq c_4 e^{-\rho t} \tag{4}$$

for all $|\theta| < \theta_0$ and $t \geq 0$. The Chapman-Kolmogorov identities, see e.g. [4, Proposition 2.12] show that

$$\frac{d}{d\theta}\Psi(\tau,\theta;\tilde{x}_0) = -\Psi(\tau,\theta;\tilde{x}_0) D_x f(\phi(\theta;\tilde{x}_0)). \tag{5}$$

Moreover, we have

$$\Psi(\tau+\theta,\theta;\tilde{x}_0) = \Psi(\tau,0;\phi(\theta;\tilde{x}_0)), \tag{6}$$

as both satisfy the initial value problem $\frac{d}{d\tau}y(\tau) = D_x f(\phi(\tau+\theta;\tilde{x}_0))y(\tau)$ with $y(0) = I$.

We define

$$M(\tilde{x}_0) = \int_0^\infty \Psi(\tau,0;\tilde{x}_0)^T C(\phi(\tau;\tilde{x}_0)) \Psi(\tau,0;\tilde{x}_0)\, d\tau.$$

We will show that M is well defined, satisfies the equation (2) and is $C^{\sigma-1}$. It is clear that $M(\tilde{x}_0)$ is then symmetric and positive definite, since C is.

We define and calculate with (6) for $|\theta| < \theta_0$

$$\begin{aligned}
g_s(\theta;\tilde{x}_0) &= \int_\theta^{s+\theta} \Psi(\tau,\theta;\tilde{x}_0)^T C(\phi(\tau;\tilde{x}_0)) \Psi(\tau,\theta;\tilde{x}_0)\, d\tau \\
&= \int_0^s \Psi(\tau+\theta,\theta;\tilde{x}_0)^T C(\phi(\tau+\theta;\tilde{x}_0)) \Psi(\tau+\theta,\theta;\tilde{x}_0)\, d\tau \\
&= \int_0^s \Psi(\tau,0;\phi(\theta;\tilde{x}_0))^T C(\phi(\tau;\phi(\theta;\tilde{x}_0))) \\
&\qquad \cdot \Psi(\tau,0;\phi(\theta;\tilde{x}_0))\, d\tau.
\end{aligned}$$

Using (4) and that C is bounded in U, we can conclude that $g_s(\theta;\tilde{x}_0)$ converges pointwise as $s \to \infty$ by Lebesgue's dominated convergence theorem. Now we compute the derivative and show that it converges uniformly. Using (5), we have

$$\begin{aligned}
&\frac{d}{d\theta} g_s(\theta;\tilde{x}_0) \\
&= \Psi(s+\theta,\theta;\tilde{x}_0)^T C(\phi(s+\theta;\tilde{x}_0)) \Psi(s+\theta,\theta;\tilde{x}_0) \\
&\quad - C(\phi(\theta;\tilde{x}_0)) - D_x f(\phi(\theta;\tilde{x}_0))^T \cdot \\
&\quad \int_\theta^{s+\theta} \Psi(\tau,\theta;\tilde{x}_0)^T C(\phi(\tau;\tilde{x}_0)) \Psi(\tau,\theta;\tilde{x}_0)\, d\tau \\
&\quad - \int_\theta^{s+\theta} \Psi(\tau,\theta;\tilde{x}_0)^T C(\phi(\tau;\tilde{x}_0)) \Psi(\tau,\theta;\tilde{x}_0)\, d\tau \cdot \\
&\quad D_x f(\phi(\theta;\tilde{x}_0)).
\end{aligned}$$

By (6) we have

$$\begin{aligned}
&\int_\theta^{s+\theta} \Psi(\tau,\theta;\tilde{x}_0)^T C(\phi(\tau;\tilde{x}_0))\Psi(\tau,\theta;\tilde{x}_0)\,d\tau\\
&= \int_0^s \Psi(\tau+\theta,\theta;\tilde{x}_0)^T C(\phi(\tau+\theta;\tilde{x}_0))\\
&\qquad\qquad \cdot\Psi(\tau+\theta,\theta;\tilde{x}_0)\,d\tau\\
&= \int_0^s \Psi(\tau,0;\phi(\theta;\tilde{x}_0))^T C(\phi(\tau;\phi(\theta;\tilde{x}_0)))\\
&\qquad\qquad \cdot\Psi(\tau,0;\phi(\theta;\tilde{x}_0))\,d\tau.
\end{aligned}$$

By (4), $\frac{d}{d\theta}g_s(\theta;\tilde{x}_0)$ converges uniformly in θ as $s\to\infty$ and

$$\begin{aligned}
\dot{M}(\tilde{x}_0) &= \frac{d}{d\theta}\lim_{s\to\infty}\int_0^s \Psi(\tau,0;\phi(\theta;\tilde{x}_0))^T\cdot\\
&\qquad C(\phi(\tau;\phi(\theta;\tilde{x}_0)))\Psi(\tau,0;\phi(\theta;\tilde{x}_0))\,d\tau\Big|_{\theta=0}\\
&= \frac{d}{d\theta}\lim_{s\to\infty} g_s(\theta;\tilde{x}_0)\Big|_{\theta=0} = \lim_{s\to\infty}\frac{d}{d\theta}g_s(\theta;\tilde{x}_0)\Big|_{\theta=0}\\
&= -C(\tilde{x}_0) - D_xf(\tilde{x}_0)^T M(\tilde{x}_0) - M(\tilde{x}_0)D_xf(\tilde{x}_0)
\end{aligned}$$

which shows that M satisfies the differential equation (2) and is well defined.

Finally, we will show that M is $C^{\sigma-1}$, adapting the argumentation of [7, Proof of Theorem 4.4, Step 3]. Denote by $-\nu<0$ the largest real part of all eigenvalues of L and let $\epsilon=\nu/2$. There is an invertible matrix $T\in\mathbb{C}^{n\times n}$ such that we have with $A=T^{-1}LT$

$$w^*\frac{1}{2}(A+A^*)w \le (-\nu+\epsilon/2)\|w\|^2 \tag{7}$$

for all $w\in\mathbb{C}^n$. Let $U\subset A(\Gamma)$ be a positively invariant, compact neighborhood of Γ such that also

$$\|T^{-1}P^{-1}(t)[D_xf(\phi(t;\tilde{x}_0)) - B(t)]P(t)T\| \le \epsilon/2 \tag{8}$$

holds for all $\tilde{x}_0\in U$ and all $t\ge 0$. Now let $\nu'=\min(\nu/4,\rho)$. We seek to show that

$$\|T^{-1}P^{-1}(t)\partial^\alpha\Psi(\tau,0;\tilde{x}_0)\| \le c_\alpha e^{-\nu' t} \tag{9}$$

holds for all $\alpha\in\mathbb{N}_0^{n+1}$ with $|\alpha|:=\sum_{i=0}^n|\alpha_i|\le\sigma-1$, $\tilde{x}_0\in U$ and $\tau\ge 0$; the 0-th derivative is with respect to t_0, and the i-th derivative with respect to $(x_0)_i$, $i\ge 1$. The proof will be by induction with respect to $k=|\alpha|$. For $k=0$, (9) follows from (4). Now we assume that (9) is true for all $|\alpha|=k-1$ where $1\le k\le\sigma-1$. Let $|\alpha'|=k\le\sigma-1$, such that $\alpha'=\alpha+e_i$ with $|\alpha|=k-1$ and $i\in\{0,1,\dots,n\}$. We have

$$\begin{aligned}
&2\|T^{-1}P^{-1}(t)\partial^{\alpha'}\Psi(t,0;\tilde{x}_0)\|\frac{d}{dt}\|T^{-1}P^{-1}(t)\partial^{\alpha'}\Psi(t,0;\tilde{x}_0)\|\\
&= \frac{d}{dt}\|T^{-1}P^{-1}(t)\partial^{\alpha'}\Psi(t,0;\tilde{x}_0)\|^2\\
&= (T^{-1}P^{-1}(t)\partial^{\alpha'}\Psi(t,0;\tilde{x}_0))^*T^{-1}\cdot\\
&\quad\left[\frac{d}{dt}[P^{-1}(t)]\partial^{\alpha'}\Psi(t,0;\tilde{x}_0) + P^{-1}(t)\frac{d}{dt}\partial^{\alpha'}\Psi(t,0;\tilde{x}_0)\right]\\
&\quad+\Bigg(T^{-1}\Big[\frac{d}{dt}[P^{-1}(t)]\partial^{\alpha'}\Psi(t,0;\tilde{x}_0)\\
&\qquad+P^{-1}(t)\frac{d}{dt}\partial^{\alpha'}\Psi(t,0;\tilde{x}_0)\Big]\Bigg)^*\cdot\\
&\qquad\quad(T^{-1}P^{-1}(t)\partial^{\alpha'}\Psi(t,0;\tilde{x}_0)).
\end{aligned}$$

Using $\dot{\Psi}(t,0;\tilde{x}_0)=D_xf(\phi(t;\tilde{x}_0))\Psi(t,0;\tilde{x}_0)$, we have

$$\begin{aligned}
&\frac{d}{dt}[P^{-1}(t)]\partial^{\alpha'}\Psi(t,0;\tilde{x}_0) + P^{-1}(t)\frac{d}{dt}\partial^{\alpha'}\Psi(t,0;\tilde{x}_0)\\
&\quad= [-P^{-1}(t)B(t)+LP^{-1}(t)]\partial^{\alpha'}\Psi(t,0;\tilde{x}_0)\\
&\qquad+P^{-1}(t)\partial^{\alpha'}D_xf(\phi(t;\tilde{x}_0))\Psi(t,0;\tilde{x}_0).
\end{aligned}$$

Similar to [7] we have

$$\begin{aligned}
&\partial^{\alpha'}D_xf(\phi(t;\tilde{x}_0))\Psi(t,0;\tilde{x}_0)\\
&\quad= D_xf(\phi(t;\tilde{x}_0))\partial^{\alpha'}\Psi(t,0;\tilde{x}_0) + r(t)
\end{aligned}$$

with $\|r(t)\|\le c_5e^{-\nu' t}$, using a generalized Leibniz rule and the induction hypothesis. Denoting $w(t)=T^{-1}P^{-1}(t)\partial^{\alpha'}\Psi(t,0;\tilde{x}_0)$, and using (7) and (8), we have

$$\begin{aligned}
\|w(t)\|\frac{d}{dt}\|w(t)\| &= w(t)^*\frac{1}{2}(T^{-1}LT+(T^{-1}LT)^*)w(t)\\
&\quad+\|w(t)\|\left[\|w(t)\|\frac{\epsilon}{2}+c_6e^{-\nu' t}\right]\\
&\le (-\nu+\epsilon)\|w(t)\|^2 + c_6e^{-\nu' t}\|w(t)\|\\
&= -2\nu'\|w(t)\|^2 + c_6e^{-\nu' t}\|w(t)\|.
\end{aligned}$$

Application of Gronwall's Lemma for $w(t)$ shows (9).

The uniform convergence of

$$\int_0^s \partial^\alpha[\Psi(\tau,0;\tilde{x}_0)^T C(\phi(\tau;\tilde{x}_0))\Psi(\tau,0;\tilde{x}_0)]\,d\tau$$

as $s\to\infty$ now follows in a similar way from (9) as in [7], which shows that $M\in C^{\sigma-1}$. The uniqueness follows similarly to the proof of [14, Theorem 2.3]. ∎

III. Meshless collocation

In this section we give a brief overview of meshless collocation to solve matrix-valued partial differential equations such as (2). We will mainly use [13], where this theory is developed, however, as our setting is time-periodic, we will also employ ideas from [12].

A. Reproducing Kernel Hilbert Space

We define a matrix-valued Reproducing Kernel Hilbert Space following [13]. Here, $\mathcal{L}(\mathbb{S}^{n\times n})$ denotes the linear space of all linear and bounded operators $L\colon\mathbb{S}^{n\times n}\to\mathbb{S}^{n\times n}$ and we use the following inner product in $\mathbb{S}^{n\times n}$ (or $\mathbb{R}^{n\times n}$): $\langle\alpha,\beta\rangle_{\mathbb{S}^{n\times n}}=\sum_{i,j=1}^n\alpha_{ij}\beta_{ij}$. We denote by $E_{\mu\nu}$ the matrix in $\mathbb{R}^{n\times n}$ with a 1 at position (μ,ν) and zero otherwise, while $E^s_{\mu\nu}=\frac{1}{\sqrt{2}}(E_{\mu\nu}+E_{\nu\mu})$ for $\mu\ne\nu$ and $E^s_{\mu\mu}=E_{\mu\mu}$. Note that $E_{\mu\nu}$ is a basis of $\mathbb{R}^{n\times n}$, while $E^s_{\mu\nu}$ is a basis of $\mathbb{S}^{n\times n}$.

Definition 4: A Hilbert space H of functions $g:S_T\times\mathbb{R}^n\to\mathbb{S}^{n\times n}$ with inner product $\langle\cdot,\cdot\rangle_H$ is called a Reproducing Kernel Hilbert space with kernel $\Phi:(S_T\times\mathbb{R}^n)\times(S_T\times\mathbb{R}^n)\to\mathcal{L}(\mathbb{S}^{n\times n})$ if

1) $\Phi(\cdot,(t,x))\alpha\in H$ for all $\alpha\in\mathbb{S}^{n\times n}$ and all $(t,x)\in S_T\times\mathbb{R}^n$,
2) $\langle g(t,x),\alpha\rangle_{\mathbb{S}^{n\times n}}=\langle g,\Phi(\cdot,(t,x))\alpha\rangle_H$ for all $g\in H$, $\alpha\in\mathbb{S}^{n\times n}$ and all $(t,x)\in S_T\times\mathbb{R}^n$.

The action of Φ on $\alpha\in\mathbb{S}^{n\times n}$ is defined by

$$(\Phi((t,x),(s,y))\alpha)_{ij} = \sum_{k,\ell=1}^n\Phi((t,x),(s,y))_{ijk\ell}\alpha_{k\ell}.$$

Note that a Hilbert space is a Reproducing Kernel Hilbert Space, iff the point evaluation functionals are continuous see [27, Theorem 10.2]. In addition, we require the kernel to be positive definite in the following sense, see [13, Definition 2.3].

Definition 5: The kernel $\Phi : (S_T \times \mathbb{R}^n) \times (S_T \times \mathbb{R}^n) \to \mathcal{L}(\mathbb{S}^{n\times n})$ is called positive definite if

$$\sum_{j,k=1}^{N} \langle \alpha_j, \Phi((t_j,x_j),(t_k,x_k))\alpha_k\rangle_{\mathbb{S}^{n\times n}} > 0$$

holds for all $N \in \mathbb{N}$, pairwise distinct points $(t_j,x_j) \in S_T \times \mathbb{R}^n$, and all $\alpha_j \in \mathbb{S}^{n\times n}$, which are not all zero, $j = 1,\dots,N$.

We use a scalar-valued kernel to define a matrix-valued reproducing kernel through

$$\Phi((t,x),(s,y))_{ij\mu\nu} = \sum_{m\in\mathbb{Z}} \phi(\|(t-s+mT, x-y)\|)\delta_{i\mu}\delta_{j\nu}, \tag{10}$$

where $\phi\colon \mathbb{R}_0^+ \to \mathbb{R}$ is a positive definite Radial Basis Function with compact support and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{n+1}$. The compact support ensures that the sum is finite. The idea is a combination of using a scalar valued kernel as in [13, Lemma 3.2], together with [12], to make it a time-periodic kernel.

Lemma 6: Let $\phi\colon \mathbb{R}_0^+ \to \mathbb{R}$ be a positive definite function, i.e. $\sum_{j,k=1}^{N} \alpha_j\alpha_k\phi(|x_j - x_k|) > 0$ for all pairwise distinct points $x_j \in \mathbb{R}$, $j = 1,\dots,N$ and all $\alpha \in \mathbb{R}^N \setminus \{0\}$.

Then the kernel defined in (10) is positive definite in the sense of Definition 5.

Proof: Firstly, $\sum_{m\in\mathbb{Z}} \phi(\|(t-s+mT, x-y)\|)$ is positive definite on $S_T\times\mathbb{R}$ by [12, Theorem 3.8]. Secondly, the matrix-valued kernel is positive definite by [13, Corollary 3.3]. ■

We will choose as ϕ the scaled Wendland function $r \mapsto \psi_{\ell,k}(cr)$, $c > 0$, with $\ell = \lfloor\frac{n+1}{2}\rfloor + k + 1$, which leads to a Hilbert space of functions, which is norm equivalent to the Sobolev space H^τ with $\tau = k+(n+2)/2$. Since $\tau > (n+1)/2$, point evaluations are continuous by the Sobolev embedding theorem and the space is a Reproducing Kernel Hilbert Space.

We will now formulate the search for the solution of (2) as an optimal recovery problem. Setting

$$\begin{aligned} F(M)(t,x) \quad := \quad & D_x f(t,x)^T M(t,x) \\ & + M(t,x)D_x f(t,x) + \dot{M}(t,x) \end{aligned} \tag{11}$$

we define the linear functionals $\lambda_k^{(i,j)}\colon H^\sigma(S_T\times\mathbb{R}^n;\mathbb{S}^{n\times n}) \to \mathbb{R}$ by

$$\lambda_k^{(i,j)}(M) \quad = \quad e_i^T F(M)(t_k,x_k)e_j =: e_i^T F_k(M)e_j \tag{12}$$

for $(t_k,x_k) \in \Omega$, $1 \le k \le N$ and $1 \le i \le j \le n$. The proof of the following theorem follows from the proof of [13, Theorem 5.2].

Theorem 7: Let $\sigma > (n+1)/2+1$, $s = \sigma+1$, $\Omega \subset S_T\times\mathbb{R}^n$ be an open set and Φ the reproducing kernel of $H^\sigma(\Omega;\mathbb{S}^{n\times n})$. Let $X = \{(t_1,x_1),\dots,(t_N,x_N)\} \subset \Omega$ be pairwise distinct points and $\lambda_k^{(i,j)}$ be defined by (12).

Then there is a unique function $S \in H^\sigma(\Omega;\mathbb{S}^{n\times n})$ which solves

$$\min\{\|S\|_{H^\sigma(\Omega;\mathbb{S}^{n\times n})}\colon \lambda_k^{(i,j)}(S) = -C_{ij}(t_k,x_k)\}$$

for all $1 \le i \le j \le n, 1 \le k \le N$. Further,

$$\begin{aligned} & S(t,x) \\ &= \sum_{k=1}^{N}\sum_{1\le i\le j\le n}\gamma_k^{(i,j)}\sum_{1\le\mu\le\nu\le n}\lambda_k^{(i,j)}(\Phi(\cdot,(t,x))E_{\mu\nu}^s)E_{\mu\nu}^s \\ &= \sum_{k=1}^{N}\sum_{i,j=1}^{n}\beta_k^{(i,j)}\sum_{\mu,\nu=1}^{n}F_k(\Phi(\cdot,(t,x))_{\cdot,\cdot,\mu,\nu})_{ij}E_{\mu\nu} \end{aligned} \tag{13}$$

where the γ_k are determined by

$$\lambda_\ell^{(i,j)}(S) \quad = \quad -C_{ij}(\tilde{x}_\ell) \tag{14}$$

for $1 \le i \le j \le N$ and $1 \le \ell \le N$, while the $\beta_k \in \mathbb{S}^{n\times n}$ are defined by $\beta_k^{(j,i)} = \beta_k^{(i,j)} = \frac{1}{2}\gamma_k^{(i,j)}$ for $i \ne j$ and $\beta_k^{(i,i)} = \gamma_k^{(i,i)}$.

Using the ideas from [12], we can also prove the following error estimate, similar to [13, Theorem 5.3].

Theorem 8: Let $f \in C^s(S_T\times\mathbb{R}^n)$, $\mathbb{N} \ni s \ge (n+1)/2+2$ and set $\sigma = s-1$. Let Γ be an exponentially stable periodic orbit of (1) with basin of attraction $A(\Gamma)$. Let $C \in C^{\sigma-1}(A(\Gamma),\mathbb{S}^{n\times n})$ be such that $C(t,x)$ is positive definite, and let $M \in C^\sigma(A(\Gamma),\mathbb{S}^{n\times n})$ be the solution of (2) from Theorem 3. Let $\Gamma \subset K \subset A(\Gamma)$ be a positively invariant, compact set, and $K \subset \Omega \subset A(\Gamma)$ be an open set with Lipschitz boundary. For S from Theorem 7 we have the error estimate

$$\begin{aligned} \|M-S\|_{L_\infty(K;\mathbb{S}^{n\times n})} &\le c_7\|F(M)-F(S)\|_{L_\infty(\Omega;\mathbb{S}^{n\times n})} \\ &\le c_8 h_{X,\Omega}^{\sigma-1-(n+1)/2}\|M\|_{H^\sigma(\Omega;\mathbb{S}^{n\times n})} \end{aligned}$$

for all $X \subset \Omega$ with sufficiently small fill distance $h_{X,\Omega} := \sup_{\tilde{x}\in\Omega}\min_{\tilde{x}_i\in X}\|\tilde{x}-\tilde{x}_i\|$, where $\|\cdot\|$ is a norm on $S_T\times\mathbb{R}^n$. In particular, S is a contraction metric if $h_{X,\Omega}$ is sufficiently small.

B. Formulas

In this section we will provide the formulas for the computation; they follow the argumentations in [14, Section 3.2]. We denote $\tilde{x} = (t,x) \in S_T\times\mathbb{R}^n$ as well as the collocation points $\tilde{x}_k = (t_k,x_k)$. We fix a Radial Basis Function with compact support, defined by $\phi(\tilde{x},\tilde{y}) = \psi_0(\|\tilde{x}-\tilde{y}\|)$, e.g. $\psi_0(r) := \psi_{\ell,k}(cr)$ as discussed above, and denote $\psi_{m+1}(r) = \frac{d\psi_m(r)/dr}{r}$ for $m = 0,1$. We assume that ψ_1 and ψ_2 can be continuously extended up to $r = 0$; this is true for (sufficiently smooth) Wendland functions. For the linear operators F_k, see (12), we have, denoting $\partial_0 := \partial_t$ and $\tilde{f}(\tilde{x}) = (f_0(\tilde{x}),\dots,f_n(\tilde{x}))^T$ with $f_0(\tilde{x}) = 1$,

$$\begin{aligned} (F_k(M))_{ij} &= \sum_{p=1}^{n} D_x f_{pi}(\tilde{x}_k)M_{pj}(\tilde{x}_k) \\ &\quad + \sum_{p=1}^{n} M_{ip}(\tilde{x}_k)D_x f_{pj}(\tilde{x}_k) + \sum_{p=0}^{n}\partial_p M_{ij}(\tilde{x}_k)f_p(\tilde{x}_k). \end{aligned}$$

In the following formulas we let

$$\tilde{v}_m := \tilde{x}_k - \tilde{x} + (mT,0) \quad \text{and} \quad r_m := \|\tilde{v}_m\|, \quad m \in \mathbb{Z};$$

that is, mT is added to the zero-th component of the vector $\tilde{x}_k - \tilde{x} \in \mathbb{R}^{n+1}$. Further, $(\tilde{v}_m)_p$ denotes the p-th component of the vector $\tilde{v}_m$. By (10) we get

$$\begin{aligned}
&(F_k(\Phi(\cdot,\tilde{x}))_{\cdot,\cdot,\mu,\nu})_{ij}\\
&= \sum_{m\in\mathbb{Z}} \Bigg[\sum_{p=1}^{n} \psi_0(r_m) D_x f_{pi}(\tilde{x}_k)\delta_{p\mu}\delta_{j\nu}\\
&\qquad + \sum_{p=1}^{n} \psi_0(r_m)\delta_{i\mu}\delta_{p\nu} D_x f_{pj}(\tilde{x}_k)\\
&\qquad + \sum_{p=0}^{n} \psi_1(r_m)(\tilde{v}_m)_p f_p(\tilde{x}_k)\delta_{i\mu}\delta_{j\nu}\Bigg]\\
&= \sum_{m\in\mathbb{Z}} \Bigg[\psi_0(r_m)[D_x f_{\mu i}(\tilde{x}_k)\delta_{j\nu} + \delta_{i\mu} D_x f_{\nu j}(\tilde{x}_k)]\\
&\qquad + \psi_1(r_m)\langle \tilde{v}_m, \tilde{f}(\tilde{x}_k)\rangle \delta_{i\mu}\delta_{j\nu}\Bigg],
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$.

Next, we compute $S(\tilde{x})$ using (13) and get

$$\begin{aligned}
S(\tilde{x}) &= \sum_{m\in\mathbb{Z}}\sum_{k=1}^{N} \Bigg[\sum_{i,\mu,\nu=1}^{n} \beta_k^{(i,\nu)} \psi_0(r_m) D_x f_{\mu i}(\tilde{x}_k) E_{\mu\nu}\\
&\qquad + \sum_{j,\mu,\nu=1}^{n} \beta_k^{(\mu,j)} \psi_0(r_m) D_x f_{\nu j}(\tilde{x}_k) E_{\mu\nu}\\
&\qquad + \sum_{\mu,\nu=1}^{n} \beta_k^{(\mu,\nu)} \psi_1(r_m)\langle \tilde{v}_m, \tilde{f}(\tilde{x}_k)\rangle E_{\mu\nu}\Bigg]\\
&= \sum_{m\in\mathbb{Z}}\sum_{k=1}^{N} \Bigg[\psi_0(r_m)\left[D_x f(\tilde{x}_k)\beta_k + \beta_k D_x f(\tilde{x}_k)^T\right]\\
&\qquad + \psi_1(r_m)\langle \tilde{v}_m, \tilde{f}(\tilde{x}_k)\rangle \beta_k\Bigg]. \qquad (15)
\end{aligned}$$

Hence,

$$\begin{aligned}
&F(S(\tilde{x})) =\\
&\sum_{m\in\mathbb{Z}}\sum_{k=1}^{N} \Bigg[\psi_0(r_m)\big[D_x f(\tilde{x})^T (D_x f(\tilde{x}_k)\beta_k + \beta_k D_x f(\tilde{x}_k)^T)\\
&\quad + (D_x f(\tilde{x}_k)\beta_k + \beta_k D_x f(\tilde{x}_k)^T) D_x f(\tilde{x})\big]\\
&\quad + \psi_1(r_m)\langle \tilde{v}_m, \tilde{f}(\tilde{x}_k)\rangle \left[D_x f(\tilde{x})^T \beta_k + \beta_k D_x f(\tilde{x})\right]\\
&\quad - \psi_1(r_m)\langle \tilde{v}_m, \tilde{f}(\tilde{x})\rangle \left[D_x f(\tilde{x}_k)\beta_k + \beta_k D_x f(\tilde{x}_k)^T\right]\\
&\quad - \psi_1(r_m)\langle \tilde{f}(\tilde{x}), \tilde{f}(\tilde{x}_k)\rangle \beta_k\\
&\quad - \psi_2(r_m)\langle \tilde{v}_m, \tilde{f}(\tilde{x}_k)\rangle\langle \tilde{v}_m, \tilde{f}(\tilde{x})\rangle \beta_k\Bigg]. \qquad (16)
\end{aligned}$$

Now we consider the linear system for the coefficients γ_k and β_k, respectively.

Let us first calculate the coefficients $b_{k,\ell,i,j,\mu,\nu}$ for $1 \le k,\ell \le N$, $1 \le i,j,\mu,\nu \le n$ such that

$$(F(S(\tilde{x}_\ell)))_{i,j} = \sum_{k=1}^{N}\sum_{\mu,\nu=1}^{n} b_{k,\ell,i,j,\mu,\nu}\beta_k^{(\mu,\nu)}, \qquad (17)$$

where $\beta_k^{(\mu,\nu)}$ is the (μ,ν)-th entry of the matrix β_k. Denoting

$$\tilde{v}_m := \tilde{x}_k - \tilde{x}_\ell + (mT, 0) \quad\text{and}\quad r_m := \|\tilde{v}_m\|, \quad m \in \mathbb{Z}$$

we have with (16)

$$\begin{aligned}
&b_{k,\ell,i,j,\mu,\nu}\\
&= \sum_{m\in\mathbb{Z}} \Bigg[\psi_0(r_m)\Bigg[\sum_{p=1}^{n} D_x f_{pi}(\tilde{x}_\ell) D_x f_{p\mu}(\tilde{x}_k)\delta_{\nu j}\\
&+ D_x f_{\mu i}(\tilde{x}_\ell) D_x f_{j\nu}(\tilde{x}_k) + D_x f_{i\mu}(\tilde{x}_k) D_x f_{\nu j}(\tilde{x}_\ell)\\
&+ \delta_{i\mu}\sum_{p=1}^{n} D_x f_{p\nu}(\tilde{x}_k) D_x f_{pj}(\tilde{x}_\ell)\Bigg]\\
&+ \psi_1(r_m)\langle \tilde{v}_m, \tilde{f}(\tilde{x}_k)\rangle \left[D_x f_{\mu i}(\tilde{x}_\ell)\delta_{\nu j} + \delta_{i\mu} D_x f_{\nu j}(\tilde{x}_\ell)\right]\\
&- \psi_1(r_m)\langle \tilde{v}_m, \tilde{f}(\tilde{x}_\ell)\rangle \left[D_x f_{i\mu}(\tilde{x}_k)\delta_{\nu j} + \delta_{i\mu} D_x f_{j\nu}(\tilde{x}_k)\right]\\
&- \psi_1(r_m)\langle \tilde{f}(\tilde{x}_\ell), \tilde{f}(\tilde{x}_k)\rangle \delta_{i\mu}\delta_{j\nu}\\
&- \psi_2(r_m)\langle \tilde{v}_m, \tilde{f}(\tilde{x}_k)\rangle \langle \tilde{v}_m, \tilde{f}(\tilde{x}_\ell)\rangle \delta_{i\mu}\delta_{j\nu}\Bigg]. \qquad (18)
\end{aligned}$$

From here, we can define the coefficients $c_{k,\ell,i,j,\mu,\nu}$ by

$$\frac{1}{4}\left(b_{k,\ell,i,j,\mu,\nu} + b_{k,\ell,j,i,\nu,\mu} + b_{k,\ell,i,j,\nu,\mu} + b_{k,\ell,j,i,\mu,\nu}\right) \qquad (19)$$

where we assume $\mu \le \nu$ and $i \le j$.

Summarising, for the computations we calculate the coefficients $c_{k,\ell,i,j,\mu,\nu}$ using (19) and (18). Then we solve

$$-C_{ij}(\tilde{x}_\ell) = \sum_{k=1}^{N}\sum_{\mu=1}^{n}\sum_{\nu=\mu}^{n} c_{k,\ell,i,j,\mu,\nu}\gamma_k^{(\mu,\nu)} \qquad (20)$$

to determine $\gamma_k^{(\mu,\nu)}$ for $i \le j$, see (14) and (17). Then, we compute $\beta_k \in \mathbb{S}^{n\times n}$ from γ_k; recall that $\beta_k^{(j,i)} = \beta_k^{(i,j)} = \frac{1}{2}\gamma_k^{(i,j)}$ if $i \ne j$ and $\beta_k^{(i,i)} = \gamma_k^{(i,i)}$. $S(\tilde{x})$ and $F(S(\tilde{x}))$ are then given by (15) and (16).

IV. Example

We apply our method to the Duffing equation, which models a mass-spring system with a hardening spring, linear viscous damping, and a periodic external force

$$m\ddot{x} + c\dot{x} + kx + ka^2x^3 = A\cos(\omega t).$$

With $y = \dot{x}$ and $c = k = m = 1$ we can write it as

$$\dot{x} = y, \quad \dot{y} = -x(1 + a^2x^2) - y + A\cos(\omega t). \qquad (21)$$

For $A = 0$ the system is autonomous and the zero solution is asymptotically stable, but for $A \ne 0$ the system is time-periodic and does not possess a stationary solution. We set $a = 0.1$, $A = 0.15$, and $\omega = 1$; note that the period of the system is $T = 2\pi$. We distributed our collocation points in $\Omega := [0, 2\pi] \times [-0.5, 0.5]^2$ using the optimal grid from [17] scaled with density parameter $\alpha = 0.35$. This resulted in $N = 252$ collocation points optimally distributed in Ω to minimize the fill-distance. As radial basis function we used the Wendland function $\psi_{4,2}$ with scaling parameter $c = 0.5$; i.e. $\psi_0(r) := \psi_{4,2}(cr)$. Checking the positive definiteness of $S(t,x,y)$ and the negative definiteness of $FS(t,x,y)$ on a dense grid around Ω showed that in Ω the numerically computed S is a contraction metric for the system. To obtain rigorous guarantees for the computed contraction metric,

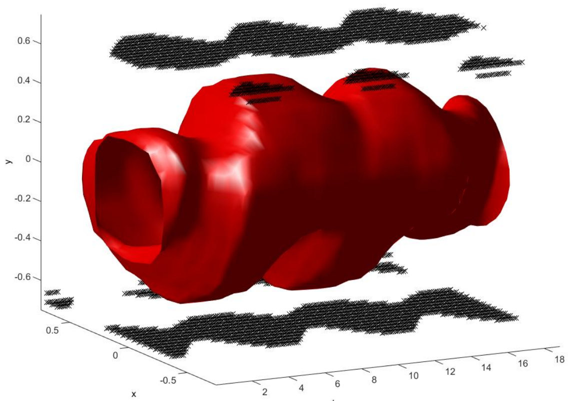

Fig. 1. Subset of the basin of attraction. The set inside the red surface is positively invariant for the dynamics of (21) with $\boldsymbol{a = 0.1}$, $\boldsymbol{A = 0.15}$, and $\boldsymbol{\omega = 1}$ [15] as a $\boldsymbol{6\pi}$-periodic system. We computed a contraction metric $\boldsymbol{S(t,x,y)}$ for time-periodic systems using the method described in the paper; the black x's at the boundary denote where $\boldsymbol{S}$ fails to fulfill the conditions of a contraction metric. Since $\boldsymbol{S(t,x,y)}$ is a contraction metric on a superset of the positively invariant set $\boldsymbol{G}$ inside the red surface, it contains exactly one periodic orbit, the periodic orbit is exponentially stable and the set is a subset of its basin of attraction.

one can interpolate the contraction metric by a continuous piecewise affine (CPA) contraction metric [11].

To apply Theorem 2, we require a positively invariant set for the system. In [15, Example 4], a positively invariant set $G \subset S_{6\pi} \times \mathbb{R}^2$ was computed, considering this system with period $T = 6\pi$. Our computed metric S is also 6π-periodic, so the set in Figure 1 satisfies the assumptions of Theorem 2 with $T = 6\pi$; in particular, there exists exactly one 6π-periodic orbit in the set and this orbit is exponentially stable, while the set is a subset of its basin of attraction.

Let $p \in \mathbb{R}^2$ be the point of the 6π-periodic orbit at time 0. Note that $P = \cup_{t\in[0,6\pi]}\phi(t;0,p) \cup \phi(t;0,\phi(2\pi;0,p))$ is a positively invariant set and satisfies the assumptions of Theorem 2 for $T = 6\pi$, so it contains a unique periodic orbit; hence, $p = \phi(2\pi;0,p)$ and the periodic orbit has in fact period $T = 2\pi$. Defining $G_i = G\big|_{[2\pi(i-1),2\pi i]\times\mathbb{R}^2}$, $i = 1,2,3$ as sets in $S_{2\pi} \times \mathbb{R}^2$, we have that the intersection $\cap_{i=1}^3 G_i$ contains the periodic orbit (of period 2π), while the union $\cup_{i=1}^3 G_i$ is a subset of its basin of attraction.

V. Conclusions

In this paper we have developed a construction method for contraction metrics for time-periodic systems. We have first shown the existence of a contraction metric, satisfying a partial differential equation. Using meshless collocation with radial basis functions, we then compute an approximate solution to this differential equation which, if sufficiently close, is itself a contraction metric.

The method can be extended to, e.g., non-autonomous systems and stochastic systems which can be viewed as perturbations of a time-periodic system by using the contraction metric computed with our proposed method [25].

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