Convex Lyapunov functions for switched discrete-time systems by linear programming

Juan Javier Palacios Roman
Department of Mechanical Engineering
Eindhoven University of Technology
Eindhoven, the Netherlands
j.j.palacios.roman@tue.nl

Sigurdur Hafstein
Faculty of Physical Sciences
University of Iceland
Reykjavik, Iceland
shafstein@hi.is

***Abstract*—We present a novel method to compute convex Lyapunov functions for discrete-time, switched systems. The individual subsystems are assumed to be linear. The method uses linear programming to construct continuous piecewise affine and convex Lyapunov functions for the switched system. We demonstrate the applicability of our method with two examples from the literature.**

***Index Terms*—Lyapunov functions, switched systems, discrete-time systems, linear programming**

I. Introduction

Switched systems have become a topic of interest in recent years. They can be used to model a wide variety of complex systems with applications in many fields, such as the control of mechanical systems, process control and automotive systems [1], [2]. The stability analysis of switched systems is however non-trivial. Much research has been done on the stability analysis of switched systems under various switching rules, both in the continuous-time and discrete-time domain.

The switching rule in switched systems can be, among other things, arbitrary or state-dependent. Several methods for the stability analysis of discrete-time switched systems under arbitrary switching have been proposed in the literature, see, e.g., [3], [4], [5], [6] and [7]. Meanwhile, discrete-time switched systems under state-dependent switching, which include discrete-time piecewise affine systems, share some similarities with general discrete-time nonlinear systems. Some methods to analyze the stability of these systems are discussed in [8], [9], [10], [11], [12] and [13].

Of particular interest is the method presented in [6], which is an extension of the method in [13] for discrete-time arbitrary switched systems. This method utilizes a triangulation of the state-space and a linear programming (LP) problem to construct a continuous piecewise affine (CPA) Lyapunov function to systems with an asymptotically stable equilibrium. The CPA Lyapunov function is defined over a subset of the basin of attraction of the equilibrium point excluding an arbitrarily small neighbourhood of the equilibrium point. The Lyapunov function shows that any solution starting in a sublevel set of the Lyapunov function will eventually enter, and remain in, the lowest sublevel set of the Lyapunov function. Even though for practical applications this result is often sufficient to meet design requirements, from a theoretical point of view it can be undesirable. Furthermore, error bounds must be included in the conditions of the LP problem, even for linear systems. Often a large number of small simplices is needed in the triangulation due to these error bounds, which causes the size of the LP problem to increase.

The main contribution of this paper is a modification of the LP problem in [6], specifically for switched linear systems, that constructs a CPA Lyapunov function over the entire domain of the state and does not need error bounds. Hence, this method can prove global asymptotic stability of the equilibrium point. This is achieved by enforcing convexity of the Lyapunov function by introducing additional linear constraints in the LP problem. For switched linear systems and convex Lyapunov functions the error bounds in [6] are namely not needed, as will be shown in Theorem 1.

The remainder of this paper is structured as follows. Firstly, the definitions used throughout this report will be introduced in Section II. Secondly, the proposed method of constructing convex CPA Lyapunov functions with linear programming will be discussed in Section III. In Section IV the proposed method will subsequently be demonstrated on an arbitrary switched system and a state-dependent switched system. Finally, concluding remarks will be given in Section V.

II. Definitions

Consider the general class of discrete-time switched linear systems

$$\mathbf{x}_{k+1} \in \{A\mathbf{x}_k : \; A \in \mathcal{A}(\mathbf{x}_k)\} \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ for all $k \in \mathbb{N}$ and $\mathcal{A} : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n\times n}$ is a set-valued function. Assume $\mathcal{A}(\mathbf{x}_k)$ is a non-empty and finite set for all $\mathbf{x}_k \in \mathbb{R}^n$ and $\mathcal{A}(a\mathbf{x}_k) = \mathcal{A}(\mathbf{x}_k)$ for all $a > 0$. A solution of (1) at time instance k with initial condition $\mathbf{x}_0$ is denoted by $\psi(k, \mathbf{x}_0)$. Note that an arbitrary switched linear system

$$\mathbf{x}_{k+1} \in \{A_1\mathbf{x}_k, A_2\mathbf{x}_k, \dots, A_M\mathbf{x}_k\}, \tag{2}$$

where $\{A_1, A_2, \dots, A_M\} \subset \mathbb{R}^{n\times n}$ with $0 < M \in \mathbb{N}$, can be described by (1) by defining $\mathcal{A}(\mathbf{x}_k) = \{A_1, A_2, \dots, A_M\}$

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for all $\mathbf{x}_k \in \mathbb{R}^n$. Similarly, a switched linear system under state-dependent switching

$$\mathbf{x}_{k+1} = A_j \mathbf{x}_k \quad \text{if } \mathbf{x}_k \in \mathcal{S}_j, \tag{3}$$

where $A_j \in \mathbb{R}^{n \times n}$ and $\mathcal{S}_j \subset \mathbb{R}^n$ is a cone, for all $j \in \{1, 2, \ldots, M\}$, can be described by (1) by defining

$$A_j \in \mathcal{A}(\mathbf{x}_k) \quad \text{if and only if } \mathbf{x}_k \in \mathcal{S}_j.$$

To study the stability of (1), the following definitions of strong global asymptotic stability (GAS) and Lyapunov functions for difference inclusions are taken from [14]. In [14] it is also proven that the existence of a Lyapunov function for (1) is equivalent to (1) being strongly GAS.

Definition 1: A difference inclusion $\mathbf{x}_{k+1} \in F(\mathbf{x}_k)$ is said to be strongly GAS if there exists a $\mathcal{KL}$-function[1] β such that for every initial condition $\mathbf{x}_0 \in \mathbb{R}^n$ all solutions $\psi(k, \mathbf{x}_0)$ satisfy

$$\|\psi(k, \mathbf{x}_0)\| \leq \beta(\|\mathbf{x}_0\|, k) \quad \forall k \in \mathbb{N},$$

where $\|\mathbf{x}\|$ can be an arbitrary norm of $\mathbf{x} \in \mathbb{R}^n$.

Definition 2: A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a Lyapunov function to the difference inclusion $\mathbf{x}_{k+1} \in F(\mathbf{x}_k)$ if there exist $\mathcal{K}_\infty$-functions α_1, α_2 and a positive definite function α_3, i.e., $\alpha_3(0) = 0$ and $\alpha_3(r) > 0$ for all $r \in \mathbb{R} \setminus \{0\}$, such that for all $\mathbf{x} \in \mathbb{R}^n$

$$\alpha_1(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|) \tag{4}$$

and

$$\sup_{\mathbf{y} \in F(\mathbf{x})} V(\mathbf{y}) - V(\mathbf{x}) \leq -\alpha_3(\|\mathbf{x}\|). \tag{5}$$

Finally, since the derivative of a continuous piecewise function $\phi(t)$ can be ill-defined, it will be necessary to use the Dini-derivative instead in those cases, which is defined as

$$D^+\phi(t) := \limsup_{h \to 0+} \frac{\phi(t+h) - \phi(t)}{h}.$$

III. Convex CPA Lyapunov functions

Our method attempts to parameterize a convex CPA Lyapunov function for (1) on a compact neighbourhood of the origin. If successful, this Lyapunov function can then be extrapolated to the whole state-space $\mathbb{R}^n$. For the parameterization, a triangulation $\mathcal{T}$ of the compact neighbourhood is required. In this section we will first discuss the specifics of the triangulation and then the construction of the CPA Lyapunov function.

A. Triangulation

The triangulation $\mathcal{T}$ of a compact subset $\mathcal{D} \subset \mathbb{R}^n$ is a partition of $\mathcal{D}$ into n-simplices. An n-simplex $\mathfrak{S}_\nu \subset \mathbb{R}^n$ with vertices $\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu \in \mathbb{R}^n$ is defined as

$$\begin{aligned} \mathfrak{S}_\nu &= \operatorname{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu) \\ &:= \left\{ \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu \colon \sum_{i=0}^n \lambda_i = 1 \ \text{ and all } \ \lambda_i \geq 0 \right\}. \end{aligned}$$

[1]We adopt the standard definitions for $\mathcal{K}$-, $\mathcal{K}_\infty$- and $\mathcal{KL}$-functions from Definitions 4.2 and 4.3 in [15].

For our application it is convenient to always have $\mathbf{x}_0^\nu = \mathbf{0}$, because we want the origin to be a vertex of all simplices.

Assume that the vectors $\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu$ are linearly independent and that the vertices of a simplex $\mathfrak{S}_\nu \in \mathcal{T}$ have a fixed order, which is why we write $\mathfrak{S}_\nu = \operatorname{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)$ rather than $\operatorname{co}\{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\}$, i.e., ordered tuple rather than set. It follows from the first assumption that a simplex $\mathfrak{S}_\nu \in \mathcal{T}$ is always non-degenerate, i.e., it has a positive n-dimensional volume, and that for every $\mathbf{x} \in \mathfrak{S}_\nu$ there is a unique set of numbers $0 \leq \lambda_i \leq 1$, $i \in \{0, 1, \ldots, n\}$, such that $\sum_{i=0}^n \lambda_i = 1$ and $\mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu$. The matrix

$$X_\nu := \begin{bmatrix} \mathbf{x}_1^\nu & \mathbf{x}_2^\nu & \ldots & \mathbf{x}_n^\nu \end{bmatrix} \in \mathbb{R}^{n \times n}$$

is invertible and well-defined for the simplex $\mathfrak{S}_\nu$ because of the first and second assumption, respectively. See [16] for a detailed discussion of $X_\nu^{-\top}$, which plays an important role in many algorithms to parameterize Lyapunov functions using linear programming.

Further, the triangulation $\mathcal{T}$ must be shape-regular, i.e., for any $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ in $\mathcal{T}$, $\nu \neq \mu$ it must hold that

$$\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \operatorname{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_k), \quad \mathbf{y}_j = \mathbf{x}_{\ell_j^\nu}^\nu = \mathbf{x}_{\ell_j^\mu}^\mu,$$

where $\mathbf{y}_0 := \mathbf{0}$, $j \in \{0, 1, \ldots, k\}$, $0 \leq k < n$, $\ell_j^\nu, \ell_j^\mu \in \{0, 1, \ldots, n\}$, and neither $\ell_j^\nu = \ell_m^\nu$ nor $\ell_j^\mu = \ell_m^\mu$ if $j \neq m$. In other words, $\mathcal{T}$ is shape-regular if any two n-simplices $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ in $\mathcal{T}$, $\nu \neq \mu$, intersect in a common lower dimensional face. Concrete triangulations fulfilling these conditions are, e.g., the triangulations $\mathcal{T}_K$ and $\mathcal{T}_K^{\mathbf{F}}$ defined in [17], which we use in our examples.

Finally, for every simplex $\mathfrak{S}_\nu \in \mathcal{T}$ a corresponding simplicial cone can be defined by

$$\mathfrak{C}_\nu := \operatorname{cone}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu) := \left\{ \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu \colon \lambda_i \geq 0 \right\}.$$

Note that since $\mathcal{T}$ is a triangulation of a neighbourhood of the origin $\mathcal{D}$, the set-theoretic union of all $\mathfrak{C}_\nu$ is necessarily equal to $\mathbb{R}^n$.

B. LP approach for CPA Lyapunov functions

Given a triangulation $\mathcal{T}$ as defined in Section III-A, a CPA function $V \colon \mathbb{R}^n \to \mathbb{R}$ can be parameterized by specifying its value at all the vertices $\mathbf{x}$ in $\mathcal{T}$, where by abuse of notation we refer to $\mathbf{x}$ as a vertex of $\mathcal{T}$ if $\mathbf{x}$ is a vertex of any simplex in $\mathcal{T}$. Let these values be denoted by $V_\mathbf{x}$, i.e., $V(\mathbf{x}) := V_\mathbf{x}$. Note that if two simplices $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$ have a common vertex $\mathbf{x}$, i.e., if $\mathbf{x} = \mathbf{x}_i^\nu = \mathbf{x}_j^\mu$ for some $i, j \in \{0, 1, \ldots, n\}$, then $V_{\mathbf{x}_i^\nu} = V_{\mathbf{x}_j^\mu}$. This ensures V is well-defined and continuous.

Note that for every $\mathbf{x} \in \mathbb{R}^n$ there exists a simplicial cone $\mathfrak{C}_\nu$ such that $\mathbf{x} \in \mathfrak{C}_\nu$, i.e., there exist $\lambda_i \geq 0$ such that $\mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu$. Therefore the CPA function V can be defined by

$$V(\mathbf{x}) := \sum_{i=0}^n \lambda_i V_{\mathbf{x}_i^\nu}. \tag{6}$$

This can equivalently be expressed as

$$V(\mathbf{x}) = \nabla V_\nu \bullet \mathbf{x} \qquad \text{if } \mathbf{x} \in \mathfrak{C}_\nu,$$

where

$$\nabla V_\nu = \mathbf{v}_\nu^\top X_\nu^{-1}$$

and

$$\mathbf{v}_\nu := \begin{bmatrix} V_{\mathbf{x}_1^\nu} & V_{\mathbf{x}_2^\nu} & \dots & V_{\mathbf{x}_n^\nu} \end{bmatrix}^\top \in \mathbb{R}^n.$$

To find suitable values for the $V_\mathbf{x}$ such that (6) is a Lyapunov function for (1), an LP problem can be used. Any feasible solution to the LP problem can then be used to parameterize a Lyapunov function for the switched system. The conditions of the LP problem are expressed by the following theorem.

Theorem 1: Consider system (1) and a triangulation $\mathcal{T}$ as in Section III-A. Let $c_1, c_2 > 0$, let $V_\mathbf{x} \in \mathbb{R}$ for every vertex $\mathbf{x}$ in $\mathcal{T}$ and let $V\colon \mathbb{R}^n \to \mathbb{R}$ be parameterized by the values $V_\mathbf{x}$ as defined above. Assume the following conditions are fulfilled:

(i) V is zero at the origin, i.e.,

$$V_\mathbf{0} = 0; \tag{7}$$

(ii) For every vertex $\mathbf{x}$ in $\mathcal{T}$

$$V_\mathbf{x} \geq c_1 \|\mathbf{x}\|; \tag{8}$$

(iii) For every simplex $\mathfrak{S}_\nu$ in $\mathcal{T}$, for every $A \in \bigcup_{\mathbf{s}\in\mathfrak{S}_\nu, \mathbf{s}\neq\mathbf{0}} \mathcal{A}(\mathbf{s})$ and for every vertex $\mathbf{x}_i^\nu$ of $\mathfrak{S}_\nu$, $i \in \{1, 2, \dots, n\}$,

$$V(A\mathbf{x}_i^\nu) - V_{\mathbf{x}_i^\nu} \leq -c_2 \|\mathbf{x}_i^\nu\|; \tag{9}$$

(iv) For every two simplices $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$ with $\mathfrak{S}_\nu \cap \mathfrak{S}_\mu \neq \{\mathbf{0}\}$, i.e., $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ have at least two common vertices, and for all $i \in \{1, 2, ..., n\}$

$$[\nabla V_\nu - \nabla V_\mu] \boldsymbol{\cdot} \mathbf{x}_i^\nu \geq 0 \text{ and } [\nabla V_\mu - \nabla V_\nu] \boldsymbol{\cdot} \mathbf{x}_i^\mu \geq 0. \tag{10}$$

Then V is a convex CPA Lyapunov function for system (1).

Remark 1: The conditions in Theorem 1 can all be formulated as linear constraints on $V_\mathbf{x}$, where $\mathbf{x}$ are vertices in $\mathcal{T}$. An LP problem with optimization variables $V_\mathbf{x}$ and constraints (i)-(iv) can thus be used to check for the existence of a convex CPA Lyapunov function.

The computational load of the resulting LP problem largely depends on the used triangulation $\mathcal{T}$. Take for example the triangulation $\mathcal{T}_K^\mathbf{F}$ from [17], where K is a measure of how refined the triangulation is. There are $p = (2K+1)^n - (2K-1)^n$ vertices $\mathbf{x}$ in $\mathcal{T}_K^\mathbf{F}$ and $N = 2^n \cdot K^{n-1} \cdot n!$ simplices in $\mathcal{T}_K^\mathbf{F}$. When using Theorem 1 and the triangulation $\mathcal{T}_K^\mathbf{F}$ to analyze the stability of an arbitrary switched linear system (2), the resulting LP problem would have p optimization variables and at least $p + MNn$ constraints, not counting constraints (iv) which vary in number. Despite not counting constraints (iv), it is already clear that the size of the LP problem scales rapidly with the refinement K and the state dimension n. This can become a computational limitation, as for some systems we might require a large K in order to construct a convex CPA Lyapunov function using the proposed method. This problem can to some extent be mitigated by introducing state transformations, as discussed in [17].

Remark 2: Note that the inequalities in (10) are trivially fulfilled for the common vertices of $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$. Hence, those constraints need not be checked in the LP problem.

Remark 3: When using Theorem 1 for a state-dependent switched linear system as defined in (3), some conservatism may be introduced. This is due to that constraint (iii) enforces

$$V(A_j\mathbf{x}) - V(\mathbf{x}) \leq -c_2 \|\mathbf{x}\|$$

for all $\mathbf{x} \in \mathfrak{C}_\nu$, where $\mathfrak{C}_\nu \cap \mathcal{S}_j \neq \{\mathbf{0}\}$, instead of for all $\mathbf{x} \in \mathfrak{C}_\nu \cap \mathcal{S}_j$. This conservatism can be reduced by refining the triangulation $\mathcal{T}$ or by choosing a different triangulation $\mathcal{T}$ such that each $\mathcal{S}_j$ is equal to the union of some $\mathfrak{C}_\nu$ in $\mathcal{T}$.

To prove Theorem 1, we first prove a lemma which shows that V is indeed a convex function.

Lemma 1: The function $V\colon \mathbb{R}^n \to \mathbb{R}$ defined in Theorem 1 is convex.

Proof: To show that $V\colon \mathbb{R}^n \to \mathbb{R}$ is convex, we show that for every $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, the function $\phi\colon [0,1] \to \mathbb{R}$ defined as

$$\phi(t) = V(\mathbf{y} + t[\mathbf{z} - \mathbf{y}]), \tag{11}$$

has a non-decreasing Dini-derivative

$$D^+\phi\colon [0, 1) \to \mathbb{R}.$$

By the Generalized Mean Value Theorem, see, e.g., [18], it then holds for an arbitrary $\alpha \in (0, 1)$ that

$$\begin{aligned} \frac{\phi(\alpha) - \phi(0)}{\alpha} &\leq \sup_{t\in[0,\alpha]} D^+\phi(t) \\ &\leq \inf_{t\in[\alpha,1)} D^+\phi(t) \leq \frac{\phi(1) - \phi(\alpha)}{1-\alpha}, \end{aligned}$$

which implies

$$\phi(\alpha) \leq (1-\alpha)\phi(0) + \alpha\phi(1). \tag{12}$$

From (11) and (12) it then follows that

$$V(\alpha\mathbf{z} + (1-\alpha)\mathbf{y}) \leq \alpha V(\mathbf{z}) + (1-\alpha)V(\mathbf{y}),$$

which shows that V is convex.

Note that it is sufficient to show that $D^+\phi$ is non-decreasing for the cases

(i) $[\mathbf{y}, \mathbf{z}] \subset \mathfrak{C}_\nu$;

(ii) and $[\mathbf{y}, \mathbf{z}] \subset \mathfrak{C}_\nu \cup \mathfrak{C}_\mu$, where $\mathbf{y} \in \mathfrak{C}_\nu \setminus \mathfrak{C}_\mu$ and $\mathbf{z} \in \mathfrak{C}_\mu \setminus \mathfrak{C}_\nu$.

Here, $[\mathbf{y}, \mathbf{z}]$ is the line segment between $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^n$, i.e.,

$$[\mathbf{y}, \mathbf{z}] := \{\alpha\mathbf{z} + (1-\alpha)\mathbf{y} \in \mathbb{R}^n :\ \alpha \in [0, 1]\}.$$

The reason it is sufficient to only consider cases (i) and (ii) is because every line segment $[\mathbf{y}, \mathbf{z}]$ can be subdivided into smaller line segments $[\mathbf{s}_i, \mathbf{s}_{i+1}]$, where $i \in \{0, 1, \dots, r-1\}$,

$$\mathbf{s}_i := s_i\mathbf{z} + (1-s_i)\mathbf{y}, \quad 0 = s_0 < s_1 < \ldots < s_r = 1$$

and

$$\bigcup_{i=1}^{r-1} [\mathbf{s}_i, \mathbf{s}_{i+1}] = [\mathbf{y}, \mathbf{z}],$$

such that each subsegment $[\mathbf{s}_i, \mathbf{s}_{i+1}]$ satisfies either the conditions of case (i) or (ii). If the Dini-derivative is non-decreasing over each such subsegment, then the Dini-derivative must also be non-decreasing over the original line segment $[\mathbf{y}, \mathbf{z}]$.

Case (i) is trivial since $V(\mathbf{x}) = \nabla V_\nu \bullet \mathbf{x}$ if $\mathbf{x} \in \mathfrak{C}_\nu$. Thus

$$D^+\phi(t) = \nabla V_\nu \bullet [\mathbf{z} - \mathbf{y}]$$

has a constant value and therefore, it is non-decreasing.

For case (ii) let $t^* \in (0,1)$ be such that

$$[\mathbf{y}, \mathbf{y} + t^*(\mathbf{z} - \mathbf{y})] \subset \mathfrak{C}_\nu$$

and

$$[\mathbf{y} + t^*(\mathbf{z} - \mathbf{y}), \mathbf{z}] \subset \mathfrak{C}_\mu.$$

Then

$$\mathbf{z}^* := \mathbf{y} + t^*(\mathbf{z} - \mathbf{y}) \in \mathfrak{C}_\nu \cap \mathfrak{C}_\mu. \tag{13}$$

Note that

$$D^+\phi(t) = \nabla V_\nu \bullet [\mathbf{z} - \mathbf{y}] \quad \text{for } t \in [0, t^*) \text{ and}$$
$$D^+\phi(t) = \nabla V_\mu \bullet [\mathbf{z} - \mathbf{y}] \quad \text{for } t \in [t^*, 1).$$

Hence, we must show that

$$\nabla V_\nu \bullet [\mathbf{z} - \mathbf{y}] \le \nabla V_\mu \bullet [\mathbf{z} - \mathbf{y}]. \tag{14}$$

Let us first consider the case where $\mathfrak{C}_\nu \cap \mathfrak{C}_\mu = \mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \{\mathbf{0}\}$. Note that then necessarily $\mathbf{z}^* = \mathbf{0}$. It follows from (13) that $\mathbf{z} = -c^*\mathbf{y}$, where $c^* = (1 - t^*)/t^* > 0$. Due to conditions (7) and (8) of the LP problem, V has a minimum at $\mathbf{0}$. Therefore, $0 < V(\mathbf{y}) = \nabla V_\nu \bullet \mathbf{y}$ and $0 < V(\mathbf{z}) = \nabla V_\mu \bullet \mathbf{z}$. Then, (14) follows from

$$\begin{aligned}\nabla V_\nu \bullet [\mathbf{z} - \mathbf{y}] &= -(1 + c^*)\nabla V_\nu \bullet \mathbf{y} \\ &< 0 < (1 + 1/c^*)\nabla V_\mu \bullet \mathbf{z} = \nabla V_\mu \bullet [\mathbf{z} - \mathbf{y}].\end{aligned}$$

Now consider the case where $\mathfrak{C}_\nu \cap \mathfrak{C}_\mu \neq \{\mathbf{0}\}$. Because the triangulation $\mathcal{T}$ is shape-regular, we have

$$\mathfrak{C}_\nu \cap \mathfrak{C}_\mu = \operatorname{cone}\{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_k\}, \quad \mathbf{y}_j = \mathbf{x}^\nu_{\ell^\nu_j} = \mathbf{x}^\mu_{\ell^\mu_j},$$

where $\mathbf{y}_0 := \mathbf{0}$, $j \in \{0, 1, \ldots, k\}$, $0 \le k < n$, $\ell^\nu_j, \ell^\mu_j \in \{0, 1, \ldots, n\}$, and neither $\ell^\nu_j = \ell^\nu_m$ nor $\ell^\mu_j = \ell^\mu_m$ if $j \neq m$. Since $\mathbf{z}^* \in \mathfrak{C}_\nu \cap \mathfrak{C}_\mu = \operatorname{cone}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_k)$, there are some constants $\lambda^*_i \ge 0$, $i \in \{0, 1, \ldots, k\}$, such that $\mathbf{z}^* = \sum_{i=0}^k \lambda^*_i \mathbf{y}_i$. Let

$$\{\mathbf{y}_{k+1}, \mathbf{y}_{k+2}, \ldots, \mathbf{y}_n\} = \{\mathbf{x}^\mu_0, \mathbf{x}^\mu_1, \ldots, \mathbf{x}^\mu_n\} \backslash \{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_k\},$$

i.e., $\mathbf{y}_{k+1}, \mathbf{y}_{k+2}, \ldots, \mathbf{y}_n$ are the vertices of $\mathfrak{S}_\mu \subset \mathfrak{C}_\mu$ that are not also vertices of $\mathfrak{S}_\nu \subset \mathfrak{C}_\nu$. Since $\mathbf{z} \in \mathfrak{C}_\mu$ there are some constants $\lambda_i \ge 0$, $i \in \{0, 1, \ldots, n\}$, such that $\mathbf{z} = \sum_{i=0}^n \lambda_i \mathbf{y}_i$. Then

$$\mathbf{z} - \mathbf{z}^* = \sum_{i=0}^k (\lambda_i - \lambda^*_i)\mathbf{y}_i + \sum_{i=k+1}^n \lambda_i \mathbf{y}_i. \tag{15}$$

Since V is continuous, it must hold that $\nabla V_\nu \bullet \mathbf{x} = \nabla V_\mu \bullet \mathbf{x}$ for all $\mathbf{x} \in \mathfrak{C}_\nu \cap \mathfrak{C}_\mu$. Therefore,

$$\nabla V_\nu \bullet \mathbf{y}_i = \nabla V_\mu \bullet \mathbf{y}_i \quad \text{for } i \in \{0, 1, \ldots, k\}. \tag{16}$$

By combining (15) and (16), we obtain

$$(\nabla V_\mu - \nabla V_\nu) \bullet [\mathbf{z} - \mathbf{z}^*] = \sum_{i=k+1}^n \lambda_i (\nabla V_\mu - \nabla V_\nu) \bullet \mathbf{y}_i \ge 0, \tag{17}$$

where the inequality in (17) follows from $\lambda_i \ge 0$ and (10). Finally, since

$$\mathbf{z} - \mathbf{z}^* = (1 - t^*)[\mathbf{z} - \mathbf{y}]$$

and $1 - t^* \ge 0$, (17) implies

$$(\nabla V_\mu - \nabla V_\nu) \bullet [\mathbf{z} - \mathbf{y}] \ge 0,$$

i.e., inequality (14), which concludes the proof. ■

With Lemma 1, Theorem 1 can finally be proven.

Proof of Theorem 1: For any $\mathbf{x} \in \mathbb{R}^n$, there is a $\mathfrak{S}_\nu \in \mathcal{T}$ such that $\mathbf{x} \in \mathfrak{C}_\nu$ and thus

$$\mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}^\nu_i \tag{18}$$

where $\lambda_i \ge 0$ for all $i \in \{0, 1, \ldots, n\}$. It then follows from (6), (7), (8) and the convexity of the norm that

$$V(\mathbf{0}) = V_\mathbf{0} = 0$$

and

$$\begin{aligned}V(\mathbf{x}) &= \sum_{i=0}^n \lambda_i V_{\mathbf{x}^\nu_i} \ge \sum_{i=0}^n \lambda_i c_1 \|\mathbf{x}^\nu_i\| \\ &\ge c_1 \left\| \sum_{i=0}^n \lambda_i \mathbf{x}^\nu_i \right\| = c_1\|\mathbf{x}\|.\end{aligned}$$

Hence $V\colon \mathbb{R}^n \to \mathbb{R}$ is positive definite and satisfies the lower bound in (4). That the upper bound in (4) is satisfied by V can trivially be shown with the Cauchy-Schwartz inequality and the equivalence of norms on $\mathbb{R}^n$.

That V satisfies (5), i.e., decreases after each timestep, follows from (6), (9), (18) and Lemma 1: For all $\mathbf{x} \in \mathfrak{S}_\nu$ and $A \in \bigcup_{\mathbf{s} \in \mathfrak{S}_\nu, \mathbf{s} \neq \mathbf{0}} \mathcal{A}(\mathbf{s})$

$$\begin{aligned}V(A\mathbf{x}) - V(\mathbf{x}) =& V\left(A \sum_{i=0}^n \lambda_i \mathbf{x}^\nu_i\right) - \sum_{i=0}^n \lambda_i V_{\mathbf{x}^\nu_i} \\ =& V\left(A \sum_{i=0}^n \lambda_i \mathbf{x}^\nu_i\right) - \sum_{i=0}^n \lambda_i V\left(A\mathbf{x}^\nu_i\right) \\ &+ \sum_{i=0}^n \lambda_i V\left(A\mathbf{x}^\nu_i\right) - \sum_{i=0}^n \lambda_i V_{\mathbf{x}^\nu_i} \\ =& \underbrace{V\left(\sum_{i=0}^n \lambda_i A\mathbf{x}^\nu_i\right) - \sum_{i=0}^n \lambda_i V\left(A\mathbf{x}^\nu_i\right)}_{\le 0 \quad \text{because } V \text{ is convex}} \\ &+ \sum_{i=0}^n \lambda_i \underbrace{\left[V\left(A\mathbf{x}^\nu_i\right) - V_{\mathbf{x}^\nu_i}\right]}_{\le -c_2\|\mathbf{x}^\nu_i\| \quad \text{from constraints (9)}} \\ \le& -c_2 \left\| \sum_{i=0}^n \lambda_i \mathbf{x}^\nu_i \right\| = -c_2\|\mathbf{x}\|.\end{aligned}$$

This result can be extended to all $\mathbf{x} \in \mathfrak{C}_\nu$ by positive homogeneity of V and (1), which shows V is indeed a Lyapunov function for (1). ■

IV. Examples

To demonstrate the method presented in this paper, two examples will be discussed. Firstly, we will consider a system with arbitrary switching as described in (2). Secondly, we will consider a system with state-dependent switching as defined in (3).

A. Arbitrary switched system

Consider the system

$$\mathbf{x}_{k+1} \in \{A\mathbf{x}_k : \ A \in \mathrm{co}(\mathcal{A})\} \tag{19}$$

where

$$\mathcal{A} = \left\{ \begin{bmatrix} 0 & 1 \\ -0.8 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -0.8 & 0.523 \end{bmatrix} \right\}.$$

This system is taken from [4], where it was shown to be robustly strongly GAS. Furthermore, in [2] it was shown that a system of the form (19) is robustly strongly GAS if and only if the arbitrary switched linear system

$$\mathbf{x}_{k+1} \in \{A\mathbf{x}_k : \ A \in \mathcal{A}\}, \tag{20}$$

is strongly GAS, which can be analyzed with Theorem 1. The phase portrait of a sample trajectory of (20) with initial condition $\begin{bmatrix} 3 & 0 \end{bmatrix}^\top$ is given in Fig. 1.

Using the method proposed in this paper, a convex CPA Lyapunov function V is constructed using the triangulation $\mathcal{T}_5^{\mathbf{F}}$ in $\mathbb{R}^2$. See, e.g., [17] for the definition of $\mathcal{T}_K^{\mathbf{F}}$. The Lyapunov function as a function of $\mathbf{x}$ is shown in Fig. 2 and the level sets of the Lyapunov function are shown in Fig. 3.

B. State-dependent switched system

Consider system (3), where $j \in \{1, 2\}$,

$$A_1 = \begin{bmatrix} 1 & 0.01 \\ -0.05 & 0.99 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0.05 \\ -0.01 & 0.99 \end{bmatrix},$$

$$S_1 = \{x \in \mathbb{R}^2 \mid x_1^2 \geq x_2^2\},$$

and

$$S_2 = \{x \in \mathbb{R}^2 \mid x_1^2 < x_2^2\}.$$

The example is taken from [8], where linear matrix inequalities are used to construct piecewise quadratic Lyapunov functions to state-dependent switched systems. In Fig. 4, the phase portrait of the trajectory of the system with initial condition $\begin{bmatrix} 3 & 0 \end{bmatrix}^\top$ is given. Note that the trajectory almost appears continuous, since the system matrices A_1 and A_2 are close to the identity matrix, i.e., the jump at each time-step is small. The regions $\mathcal{S}_1$ and $\mathcal{S}_2$ are also shown in Fig. 4.

Using Theorem 1, a convex CPA Lyapunov function V can be constructed for the example using the triangulation $\mathcal{T}_5^{\mathbf{F}}$ in $\mathbb{R}^2$. The Lyapunov function as a function of $\mathbf{x}$ is shown in Fig. 5 and the level sets of the Lyapunov function are shown in Fig. 6.

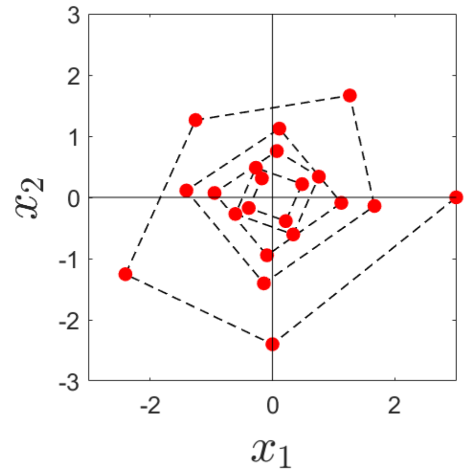

Fig. 1. Phase portrait of a trajectory of the arbitrary switched system.

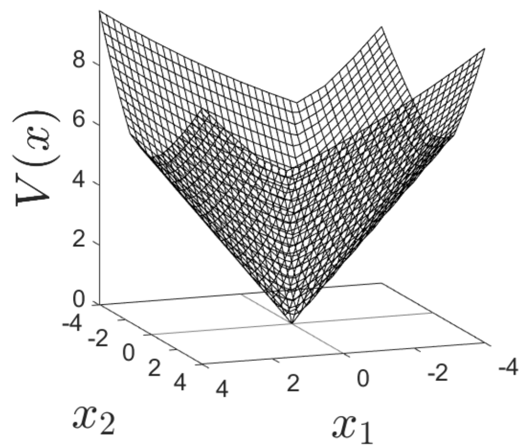

Fig. 2. A convex CPA Lyapunov function of the arbitrary switched system.

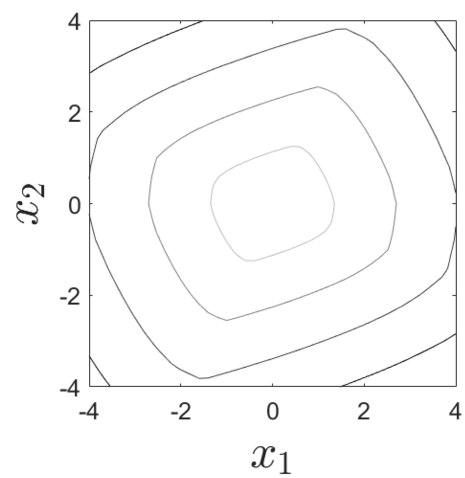

Fig. 3. Level sets of the convex CPA Lyapunov function of the arbitrary switched system.

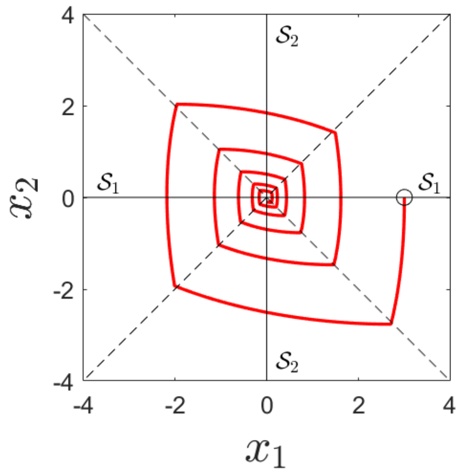

Fig. 4. Phase portrait of a trajectory of the state-dependent switched system.

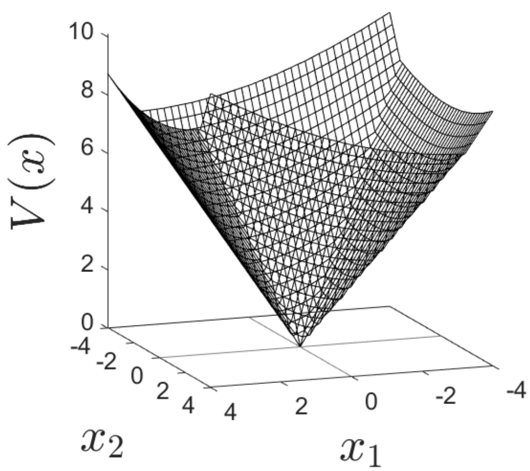

Fig. 5. A convex CPA Lyapunov function of the state-dependent switched system.

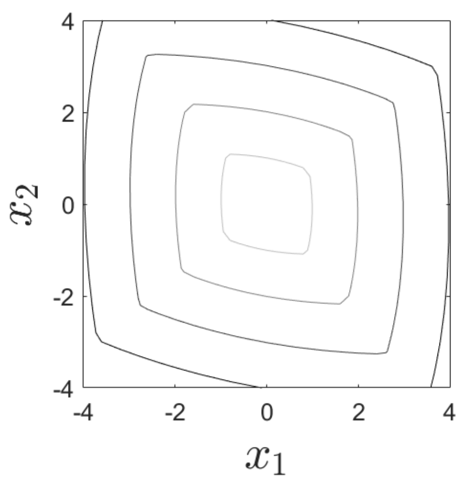

Fig. 6. Level sets of the convex CPA Lyapunov function of the state-dependent switched system.

V. Conclusion

In this paper, we have shown an LP problem that can construct a convex CPA Lyapunov function for discrete-time switched linear systems under arbitrary or state-dependent switching with a globally asymptotically stable equilibrium. The CPA Lyapunov function is defined over the entire state space and thus proves global asymptotic stability of the origin, which is an extension of the results in [6]. It was also shown that, unlike [6], this method does not require the inclusion of error bounds in the constraints of the LP problem, which reduces the required refinement of the triangulation used to construct the Lyapunov function.

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