Stability Analysis of Hybrid Integrator-Gain Systems using Linear Programming ⋆

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Abstract: Stability and performance of hybrid integrator-gain systems (HIGS) are mostly analyzed using linear matrix inequalities (LMIs) to construct continuous piecewise quadratic (CPQ) Lyapunov functions. To create additional methods for the analysis of HIGS and other conewise linear systems, a method based on linear programming (LP) for constructing continuous piecewise affine (CPA) Lyapunov functions is investigated. In this paper, it is shown how linear programming can be used to prove input-to-state stability and calculate upper bounds on the L_1-gain and H_1-norm. The numerical efficiency of this CPA (LP) method will be compared to that of the CPQ (LMI) method on a numerical example involving HIGS.

Keywords: stability of hybrid systems, input-to-state stability, linear programming, Lyapunov methods, performance analysis.

1. INTRODUCTION

In order to meet the ever increasing demand for precision and throughput in high-tech motion systems, controllers with progressively stricter performance requirements need to be designed. Although linear time-invariant (LTI) controllers are achieving high accuracy in many applications, they suffer from inherent limitations in performance (Seron et al., 2011). Therefore, nonlinear controllers have become a topic of interest, since they can potentially overcome these limitations for LTI plants.

One such class of nonlinear controllers are hybrid controllers, which includes, among others, reset control (Clegg, 1958; Zaccarian et al., 2005) and hybrid integrator-gain systems (HIGS) (Deenen et al., 2017). When evaluated from a frequency domain approximation, these controllers behave similarly to an LTI integrator, in that they induce a 20 dB/decade amplitude decay, but only induce 38.15° phase lag (as opposed to 90° phase lag by an LTI integrator). As such, these methods can potentially achieve better performance than the LTI integrator. While reset control uses discontinuous control actions, HIGS provides a continuous control input, potentially making it more suited for systems with higher-order dynamics and allowing for easier analysis. HIGS-controlled systems have been shown to be well-posed (Deenen et al., 2021; Heemels and Tanwani, 2023) and to achieve better performance in high-precision motion systems where previously LTI integrators were used (van den Eijnden et al., 2020; Shi et al., 2023).

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A disadvantage of HIGS compared to LTI integrators is, however, that the tools for stability and performance analysis are more limited. An LTI plant in a feedback loop with HIGS is namely transformed into a discontinuous conewise linear system (Camlibel et al., 2006), where some of the subdynamics are only active on a lower dimensional subset of the state space. This causes frequency domain-based approaches for guaranteeing stability to be no longer directly and non-conservatively applicable. Thus, alternate methods based on linear matrix inequalities (LMIs) have been proposed for guaranteeing input-to-state stability (ISS) in Deenen et al. (2021) and van den Eijnden et al. (2022). In these methods, common quadratic (CQ) Lyapunov functions and continuous piecewise quadratic (CPQ) Lyapunov functions (Johansson and Rantzer, 1998) are constructed to prove ISS. These methods can be extended to calculate upper bounds on the H_2-norm and L_2-gain of conewise linear systems, which are commonly used performance metrics (Aangenent, 2008; van den Eijnden et al., 2022).

An alternative method for analyzing the stability of both linear and nonlinear systems is to use linear programming (LP) to construct continuous piecewise affine (CPA) Lyapunov functions. Although not used for HIGS and its variations so far, this method has already been well discussed in the literature and several ways of formulating the LP problem have been proposed, see, e.g., Polański (1997, 2000), Blanchini (1994, 1995), Julian et al. (1999), Ohta and Yamamoto (2000), Ohta and Tsuji (2003), Hafstein (2002), Johansson (2002), Milani (2004), Yfoulis and Shorten (2004), Lazar and Doban (2011), Baier et al. (2019), Samanipour and Poonawala (2023) and Andersen et al. (2023). These existing results motivate to study this CPA method for the specific class of HIGS-controlled systems, which we will do in this paper.

We present three main contributions. Firstly, a theorem for constructing CPA Lyapunov functions for HIGS-controlled systems and general conewise linear system will be presented. The theorem is a modification of the CPA method from Andersen et al. (2023) with regional relaxations in the conditions of the LP problem. Secondly, it will be shown that the presented theorem can also be extended to calculate upper bounds on the L_1-gain and H_1-norm of the system, which can be used as performance metrics. Finally, the proposed method will be compared to the CPQ method using LMIs in a numerical case study.

The remainder of this paper is structured as follows. Section 2 introduces the notations and definitions used throughout this paper. In Section 3 HIGS is discussed. Section 4 presents the modified CPA method for the stability analysis of conewise linear systems and for the calculation of upper bounds on the L_1-gain and H_1-norm. In Section 5 the CPA method will be compared to the CPQ method in a numerical case study. Finally, concluding remarks are given in Section 6.

2. NOTATION AND DEFINITIONS

In this paper, the subset of $\mathbb{R}^n$ with nonnegative entries is denoted by $\mathbb{R}^n_{\geq 0}$. Inequalities for vectors hold componentwise. The 1-norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$, where x_i is the ith component of x. Unless otherwise specified, $\|\cdot\|$ can be an arbitrary norm.

Definition 1. A signal $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is piecewise continuous (denoted by $x \in PC$) if there exists $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_{\geq 0}$, where $t_0 = 0$, $t_{k+1} > t_k$ for all $k \in \mathbb{N}$, $\lim_{k \to \infty} t_k = \infty$, x is continuous for all $t \in \mathbb{R}_{\geq 0} \setminus \{t_k\}_{k \in \mathbb{N}}$ and $\lim_{t \to t_k^+} x(t) = x(t_k)$. If $\int_0^\infty \|x(t)\|_1 \,\mathrm{d}t < \infty$ in addition to $x \in PC$, then we write $x \in PC_1$.

Consider a (possibly discontinuous) conewise linear system

$$\begin{aligned} \dot{x}(t) &= A_j x(t) + B w(t) \quad \text{if } x(t) \in \mathcal{X}_j \\ y(t) &= C x(t) + D w(t) \end{aligned} \tag{1}$$

with state $x(t) \in \mathbb{R}^n$, disturbances $w(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$, all at time $t \in \mathbb{R}_{\geq 0}$, real matrices of appropriate dimensions A_j, B, C and D, and subsets $\mathcal{X}_j \subset \mathbb{R}^n$, where $j \in \{1, ..., M\}$, $M \in \mathbb{N}$. The closure of $\mathcal{X}_j$ is described by either a simplicial cone or the intersection of a simplicial cone and a hyperplane, i.e.,

$$\bar{\mathcal{X}}_j = \{x \in \mathbb{R}^n \mid F_j x \geq 0 \wedge \Pi_j x = 0\},$$

where $F_j \in \mathbb{R}^{n \times n}$ is an invertible matrix and $\Pi_j^\top \in \mathbb{R}^n$ is an n-dimensional row vector. The signal $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a solution to (1) in the sense of Carathéodory, if x is absolutely continuous and (1) is satisfied for almost all $t \in \mathbb{R}_{\geq 0}$ (Walter, 1998). Existence and forward completeness of solutions is assumed, which for HIGS-controlled systems can be proven if $w \in PC$ (Heemels and Tanwani, 2023). Since the derivative of a solution x can be ill-defined, the Dini-derivative is used instead in those cases, defined by

$$D^+ x(t) = \limsup_{h \to 0^+} \frac{x(t+h) - x(t)}{h}.$$

To make the notion of stability of discontinuous systems precise, we introduce the following definitions and theorem from Heemels and Weiland (2008), which are based on the work of Sontag (1989).

Definition 2. System (1) is said to be input-to-state stable (ISS), if there exist a $\mathcal{KL}$-function [1] β and a $\mathcal{K}$-function γ, such that for any initial state $x(0) \in \mathbb{R}^n$ and any bounded input $w \in PC$, any corresponding solution x of (1) satisfies

$$\|x(t)\| \leq \beta\left(\|x(0)\|, t\right) + \gamma\left(\sup_{0 \leq \tau \leq t} \|w(\tau)\|\right) \quad \forall t \in \mathbb{R}_{\geq 0}.$$

Definition 3. A locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be an ISS Lyapunov function for system (1), if it satisfies for all solutions x and corresponding $w \in PC$

$$\alpha_1(||x(t)||) \leq V(x(t)) \leq \alpha_2(||x(t)||) \quad \text{and} \tag{2a}$$

$$D^+ V(x(t)) \leq -\alpha_3(||x(t)||) + \alpha_4(\|w(t)\|) \tag{2b}$$

for all $t \in \mathbb{R}_{\geq 0}$, where α_1, α_2 and α_3 are $\mathcal{K}_\infty$-functions and α_4 is a $\mathcal{K}$-function.

Theorem 4. If there exists an ISS Lyapunov function for (1), then (1) is ISS.

Finally, consider the following performance metrics.

Definition 5. (Ebihara et al., 2011) The L_1-gain, i.e., the L_1-induced norm, of (1) is defined as the infimal value of γ for which

$$\int_0^\infty \|y(t)\|_1 \,\mathrm{d}t \leq \gamma \int_0^\infty \|w(t)\|_1 \,\mathrm{d}t$$

for all $w \in PC_1$, $x(0) = 0$ and any corresponding output y.

Definition 6. The H_1-norm of system (1) corresponding to an initial value $x(0) = x_0$ is defined as the infimal value of δ for which

$$\int_0^\infty \|y(t)\|_1 \,\mathrm{d}t \leq \delta,$$

for any output y given initial condition $x(0) = x_0$ and $w(t) = 0$ for all $t \in \mathbb{R}_{\geq 0}$.

The H_1-norm is defined in line with the H_2-norm of hybrid systems (Aangenent, 2008; van den Eijnden et al., 2022).

3. HYBRID-INTEGRATOR GAIN SYSTEMS

In continuous-time, HIGS can be described by

$$\mathcal{H} : \begin{cases} \dot{x}_h(t) = \omega_h e(t) & \text{if } (e(t), u(t), \dot{e}(t)) \in \mathcal{F}_1 \quad \text{(3a)} \\ x_h(t) = k_h e(t) & \text{if } (e(t), u(t), \dot{e}(t)) \in \mathcal{F}_2 \quad \text{(3b)} \\ u(t) \;= x_h(t) \end{cases}$$

with state $x_h(t) \in \mathbb{R}$, error $e(t) \in \mathbb{R}$ and its derivative $\dot{e}(t) \in \mathbb{R}$, control action $u(t) \in \mathbb{R}$, all at time $t \in \mathbb{R}_{\geq 0}$, parameters ω_h, $k_h \in [0, \infty)$ and subsets $\mathcal{F}_1$, $\mathcal{F}_2 \subset \mathbb{R}^3$. The mode (3a) is called the integrator mode and the mode (3b) the gain mode. The subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined by

$$\begin{aligned} \mathcal{F} &:= \mathcal{F}_1 \cup \mathcal{F}_2 = \{(e, u, \dot{e}) \in \mathbb{R}^3 \mid k_h e u \geq u^2\} \\ \mathcal{F}_1 &:= \mathcal{F} \setminus \mathcal{F}_2 \\ \mathcal{F}_2 &:= \{(e, u, \dot{e}) \in \mathbb{R}^3 \mid u = k_h e \wedge \omega_h e^2 > k_h e \dot{e}\}. \end{aligned}$$

The HIGS can be interpreted as follows: Assume that $(e, u, \dot{e}) \in \mathcal{F}$ at $t = 0$, e.g., if $x_h(0) = 0$. Then, standardly, the integrator mode is active. However, if the integrator dynamics would cause $(e, u, \dot{e})$ to leave the set $\mathcal{F}$, then

[1] We adopt the standard definitions for $\mathcal{K}$-, $\mathcal{K}_\infty$- and $\mathcal{KL}$-functions from Definitions 4.2 and 4.3 in Khalil (2002).

the gain mode is momentarily activated. This ensures that $(e, u, \dot{e})$ lies in $\mathcal{F}$ for all $t \in \mathbb{R}_{\geq 0}$.

The HIGS controller above is placed in a negative feedback loop with the LTI system $\mathcal{G}$ given by

$$\mathcal{G}: \begin{cases} \dot{x}_g(t) = A_g x_g(t) + B_{gu} u(t) + B_{gw} w(t) \\ e(t) \;\; = C_{ge} x_g(t) + D_{geu} u(t) + D_{gew} w(t) \\ y(t) \;\; = C_{gy} x_g(t) + D_{gyu} u(t) + D_{gyw} w(t) \end{cases}$$

with state $x_g(t) \in \mathbb{R}^{n_g}$, control action $u(t) \in \mathbb{R}$, disturbances $w(t) \in \mathbb{R}^m$, error $e \in \mathbb{R}$, output $y \in \mathbb{R}^p$, all at time $t \in \mathbb{R}_{\geq 0}$, and matrices A_g, B_{gu}, B_{gw}, C_{ge}, C_{gy}, D_{geu}, D_{gyu}, D_{gew} and D_{gyw} of appropriate dimensions. The system $\mathcal{G}$ includes plant dynamics and potentially LTI controllers present in the control loop. Provided that the transformations from u and w to e in $\mathcal{G}$ have relative degree two or higher, the HIGS-controlled system can be described by

$$\begin{aligned} \dot{x}(t) &= \begin{cases} A_1 x(t) + Bw(t) & \text{if } Hx(t) \in \mathcal{F}_1 \\ A_2 x(t) + Bw(t) & \text{if } Hx(t) \in \mathcal{F}_2 \end{cases} \\ y(t) &= Cx(t) + Dw(t), \end{aligned} \tag{4}$$

where $x = \begin{bmatrix} x_g^\top & x_h \end{bmatrix}^\top \in \mathbb{R}^n$ with $n = n_g + 1$,

$$\left[\begin{array}{c|c} A_k & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A_g & -B_{gu} & B_{gw} \\ A_{h,k} & 0 & 0_{1\times m} \\ \hline C_{gy} & -D_{gyu} & D_{gyw} \end{array}\right],$$

$$A_{h,1} = \omega_h C_{ge}, \quad A_{h,2} = k_h C_{ge} A_g$$

and

$$H = \begin{bmatrix} C_{ge} & 0 \\ 0_{1\times n_g} & 1 \\ C_{ge} A_g & 0 \end{bmatrix}.$$

Note that (4) can be written as a conewise linear system (1), since regions $\mathcal{F}_1$ and $\mathcal{F}_2$ can be partitioned into subregions $\mathcal{X}_j$ that satisfy (2).

4. STABILITY & PERFORMANCE ANALYSIS WITH LINEAR PROGRAMMING

In order to guarantee ISS of (4), an ISS Lyapunov function is parameterized using linear programming. This parameterization makes use of a triangulation of the domain of the state space. In this section, the specifics of the triangulation will be discussed and it will be shown how to construct a CPA Lyapunov function with linear programming. To show that this method can be applied to a broader class of systems than just the HIGS-controlled system (4), we will consider the general class of conewise linear systems (1).

4.1 The triangulation

Typically, a triangulation is a set of simplices, but for the purpose of this paper it is more convenient to use simplicial cones as the state equation of system (1) is homogeneous of degree one. A simplicial cone $\mathcal{S}_i \subset \mathbb{R}^n$ is the positive hull of n linearly independent vertices $r_1^i, ..., r_n^i$, i.e.,

$$\begin{aligned} \mathcal{S}_i &= \left\{ x \in \mathbb{R}^n \mid x = \sum_{k=1}^{n} \lambda_k r_k^i \text{ for some } \lambda_k \geq 0 \right\} \\ &= \{x \in \mathbb{R}^n \mid E_i x \geq 0\}, \end{aligned}$$

where $E_i := \begin{bmatrix} r_1^i & \dots & r_n^i \end{bmatrix}^{-1}$.

Let $\mathcal{X} := \bigcup_{j=1}^{M} \mathcal{X}_j$ be the domain of the state x. Then, a triangulation $\mathcal{T}$ of $\mathcal{X}$ is a partitioning of $\mathcal{X}$ into a collection of simplicial cones $\{\mathcal{S}_1, ..., \mathcal{S}_N\}$, $N \in \mathbb{N}$, that satisfies $\mathcal{X} = \bigcup_{i=1}^{N} \mathcal{S}_i$. The triangulation $\mathcal{T}$ is shape-regular if all $\mathcal{S}_i$ and $\mathcal{S}_j$, where $i, j \in \{1, ..., N\}$ and $i \neq j$, either intersect in a common face or not at all. An example of a shape-regular triangulation is the triangulation $\mathcal{T}_K^F$ from Andersen et al. (2023), where $K \in \mathbb{R}^n$ is a measure of how refined the triangulation is. We will use the triangulation $\mathcal{T}_K^F$ for the numerical case study in Section 5.

4.2 Construction of CPA Lyapunov functions

To parameterize a CPA function using a triangulation $\mathcal{T}$ of $\mathcal{X}$, a value V_r is assigned to each unique vertex r of a simplicial cone in $\mathcal{T}$. Then, a CPA Lyapunov function candidate can be described by

$$V(x) = v_i^\top E_i x \quad \text{ if } x \in \mathcal{S}_i, \tag{5}$$

where $v_i^\top = \begin{bmatrix} V_{r_1^i} & \dots & V_{r_n^i} \end{bmatrix}$, $i \in \{1, ..., N\}$ and the dependency of $x(t)$, $w(t)$ and $y(t)$ on t is from here on omitted for brevity where possible. Note that V is positively homogeneous of degree one, V is locally Lipschitz continuous by construction, $V(r_k^i) = V_{r_k^i}$ for all $k \in \{1, ..., n\}$ and the value of $V(x)$ in (5) is found by linearly interpolating between the values of $V(r_k^i)$ and $V(0) = 0$.

The following theorem poses conditions under which (5) is an ISS Lyapunov function.

Theorem 7. Consider system (1) and a shape-regular triangulation $\mathcal{T}$ of $\mathcal{X}$. Suppose there exist variables V_r, γ, $\delta \in \mathbb{R}$, nonnegative vectors $u_{ij} \in \mathbb{R}_{\geq 0}^n$, real variables $l_{ij} \in \mathbb{R}$, where $i \in \{1, ..., N\}$ and $j \in \{1, ..., M\}$, and constants $\epsilon_1, \epsilon_2 > 0$ such that:

(i) for every vertex r of a simplicial cone in $\mathcal{T}$
$$V_r \geq \epsilon_1 \|r\|;$$
(ii) for all $i \in \{1, ..., N\}$, $j \in \{1, ..., M\}$ and $k \in \{1, ..., n\}$
$$(v_i^\top E_i A_j + u_{ij}^\top F_j + l_{ij} \Pi_j) r_k^i \leq -\epsilon_2 \left\| r_k^i \right\| - \left\| C r_k^i \right\|_1;$$
(iii) for all $i \in \{1, ..., N\}$ and $l \in \{1, ..., m\}$
$$\pm v_i^\top E_i B e_l \leq \gamma \|e_l\|_1 - \|De_l\|_1,$$
where e_l is the lth standard unit vector of $\mathbb{R}^m$;
(iv) given $x_0 \in \mathcal{S}_q$
$$v_q^\top E_q x_0 \leq \delta.$$

Then (5) is an ISS Lyapunov function of (1). Additionally, γ is an upper bound on the L_1-gain of (1) and δ is an upper bound on the H_1-norm corresponding to an initial condition $x(0) = x_0$ of (1).

Proof. Firstly, it is proven that V satisfies condition (2a). Because of (i) and the triangle inequality, V satisfies

$$\begin{aligned} V(x) &= v_i^\top E_i x = v_i^\top E_i \sum_{k=1}^{n} \lambda_k r_k^i = \sum_{k=1}^{n} \lambda_k (v_i^\top E_i r_k^i) \\ &= \sum_{k=1}^{n} \lambda_k V_{r_k^i} \geq \sum_{k=1}^{n} \lambda_k \epsilon_1 \left\| r_k^i \right\| = \epsilon_1 \sum_{k=1}^{n} \lambda_k \left\| r_k^i \right\| \\ &\geq \epsilon_1 \left\| \sum_{k=1}^{n} \lambda_k r_k^i \right\| = \epsilon_1 \|x\|. \end{aligned}$$

Moreover, from the Cauchy-Schwarz inequality and the equivalence of norms it follows that

$$V(x) = v_i^\top E_i x = E_i^\top v_i \bullet x \leq \left| E_i^\top v_i \bullet x \right|$$
$$\leq \left\| E_i^\top v_i \right\|_2 \|x\|_2 \leq c_1 \|x\|$$

for some positive constant c_1. Thus V satisfies (2a).

Secondly, it is proven that V satisfies condition (2b). Note that, if $x(t) \in \mathcal{S}_i$ and $x(t+\delta t) \in \mathcal{S}_i$ for an infinitesimal small time step δt, then

$$D^+ V(x) = v_i^\top E_i (A_j x + Bw) \qquad \forall x \in \mathcal{X}_j.$$

Furthermore, note that by Farkas' lemma and the Fredholm alternative, the condition

$$v_i^\top E_i (A_j x + Bw) < -c_2 \|x\| + c_3 \|w\| \quad \forall x \in \bar{\mathcal{X}}_j \backslash \{0\}$$

for some $c_2, c_3 > 0$ and all $w \in \mathbb{R}^m$, is equivalent to

$$\begin{aligned} v_i^\top E_i (A_j x + Bw) + u_{ij}^\top F_j x + l_{ij} \Pi_j x < \\ - c_2 \|x\| + c_3 \|w\| \quad \forall x \in \mathbb{R}^n \backslash \{0\}. \end{aligned} \tag{6}$$

Thus, by proving (6), we can prove V satisfies condition (2b). Since $x \in \mathcal{S}_i$ for some $i \in \{1, ..., N\}$, $x = \sum_{k=1}^{n} \lambda_k r_k^i$ for some $\lambda_k \geq 0$. Similarly, $w = \sum_{l=1}^{m} \mu_l e_l$ for some real variables μ_l. Thus, (ii) and (iii) imply that

$$\begin{aligned} D^+ V(x) &\leq (v_i^\top E_i A_j + u_{ij}^\top F_j + l_{ij} \Pi_j) x + v_i^\top E_i B w \\ &= (v_i^\top E_i A_j + u_{ij}^\top F_j + l_{ij} \Pi_j) \sum_{k=1}^{n} \lambda_k r_k^i + v_i^\top E_i B \sum_{l=1}^{m} \mu_l e_l \\ &= \sum_{k=1}^{n} \lambda_k (v_i^\top E_i A_j + u_{ij}^\top F_j + l_{ij} \Pi_j) r_k^i + \sum_{l=1}^{m} \mu_l v_i^\top E_i B e_l \\ &\leq \sum_{k=1}^{n} \lambda_k (-\epsilon_2 \left\| r_k^i \right\| - \left\| C r_k^i \right\|_1) + \sum_{l=1}^{m} |\mu_l| \left(\gamma \|e_l\|_1 - \|D e_l\|_1 \right) \\ &\leq -\epsilon_2 \|x\| - \|Cx\|_1 - \|Dw\|_1 + \gamma \|w\|_1 \qquad (7) \\ &\leq -\epsilon_2 \|x\| - \|y\|_1 + \gamma \|w\|_1 . \qquad (8) \end{aligned}$$

Eq. (7) proves (6) and therefore V satisfies (2b) and V is an ISS Lyapunov function of (1).

Thirdly, it is shown that γ is an upperbound on the L_1-gain. By integrating both sides of (8) from 0 to $T > 0$, we obtain that

$$\begin{aligned} &\int_0^T D^+ V(x(t)) \mathrm{d}t = V(x(T)) - V(x(0)) \\ &\leq -\epsilon_2 \int_0^T \left\| x(t) \right\| \mathrm{d}t - \int_0^T \left\| y(t) \right\|_1 \mathrm{d}t + \gamma \int_0^T \left\| w(t) \right\|_1 \mathrm{d}t. \end{aligned} \tag{9}$$

Since V is a positive definite function and given $x(0) = 0$ (see Definition 5), this implies that

$$\gamma \int_0^T \left\| w(t) \right\|_1 \mathrm{d}t \geq \int_0^T \left\| y(t) \right\|_1 \mathrm{d}t$$

for all $T > 0$, which proves γ is an upper bound on the L_1-gain of (1) when $T \to \infty$.

Finally, we will prove that δ is an upper bound on the H_1-norm corresponding to an initial condition $x(0) = x_0$. Since $w(t) = 0 \; \forall t \in \mathbb{R}_{\geq 0}$ (see Definition 6) and therefore $\lim_{T\to\infty} V(x(T)) = 0$ (because the system is ISS), it follows from (iv) and (9) that

$$\delta \geq v_q^\top E_q x_0 = V(x(0)) \geq \int_0^\infty \left\| y(t) \right\|_1 ,$$

which proves δ is an upper bound on the H_1-norm corresponding to initial condition x_0 of (1). □

Note that Theorem 7 is a feasibility problem, which can be expressed as a linear programming problem with optimization variables V_r, γ, δ, u_{ij} and l_{ij}, constants ϵ_1 and ϵ_2 and linear constraints (i)-(iv).

Remark 8. Systems with discontinuous state equations such as (4) can have robust stability issues (Goebel et al., 2012). To obtain robust stability guarantees, the stability of the Krasovskii regularization of (4), which is defined as

$$\begin{aligned} \dot{x}(t) &\in \begin{cases} A_1 x(t) + Bw(t) & \text{if } Hx(t) \in \mathcal{F}_1 \backslash \bar{\mathcal{F}}_2 \\ \mathrm{co}(A_1 x(t), A_2 x(t)) + Bw(t) & \text{if } Hx(t) \in \bar{\mathcal{F}}_2 \end{cases} \\ e(t) &= Cx(t) + Dw(t), \end{aligned}$$

where $\mathrm{co}(\mathcal{F})$ denotes the closed convex hull of a set $\mathcal{F} \subset \mathbb{R}^n$, should be studied. It can be shown that the conditions in Theorem 7 can guarantee stability of the Krasovskii regularized system as well.

5. NUMERICAL CASE STUDY

In this section, the CPA method will be used to analyze the stability of a HIGS-controlled system and it will be compared to the method in van den Eijnden et al. (2022). This method, based on the work of Johansson and Rantzer (1998), amounts to a set of linear matrix inequalities (LMIs), which, if feasible, prove the existence of a continuous piecewise quadratic (CPQ) Lyapunov function. To solve the LP and LMI problems, the solver Mosek (2023) implemented in MATLAB with Yalmip (2004) is used.

5.1 System description

Consider the LTI mass-spring-damper system

$$\mathcal{G}: \begin{cases} \dot{x}_g(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} x_g(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) \\ e(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_g(t) \\ y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_g(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \end{cases}$$

where $m = b = k = 1$. The LTI system is placed in a negative feedback loop with HIGS as discussed in Section 3. The resulting HIGS-controlled system can be described by (4) or (1).

It is discussed by Andersen et al. (2023) that it is beneficial to the CPA method to introduce state transformations that cause the level sets of Lyapunov functions to resemble hyperspheres. The reason is that it reduces the required refinement of the triangulation $\mathcal{T}_K^F$. For HIGS-controlled systems specifically, the fact that the integrator mode is the dominant mode of the system can be exploited by introducing the state transformation

$$\hat{x} = V^{-1} x,$$

where the columns of V are the real and, if applicable, imaginary parts of the normalized eigenvectors of A_1, not counting complex conjugates. We make use of this transformation in the following results.

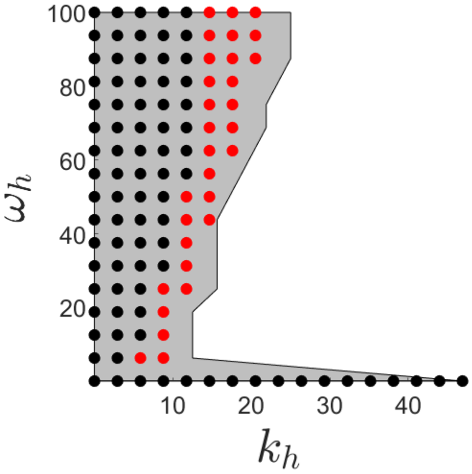

Fig. 1. ISS region found using time-series simulations (grey), ISS combinations (k_h, ω_h) found using the CPA method with triangulation $\mathcal{T}_5^F$ (black) and ISS combinations (k_h, ω_h) found using the CPQ method with triangulation $\mathcal{T}_2^F$ (black and red).

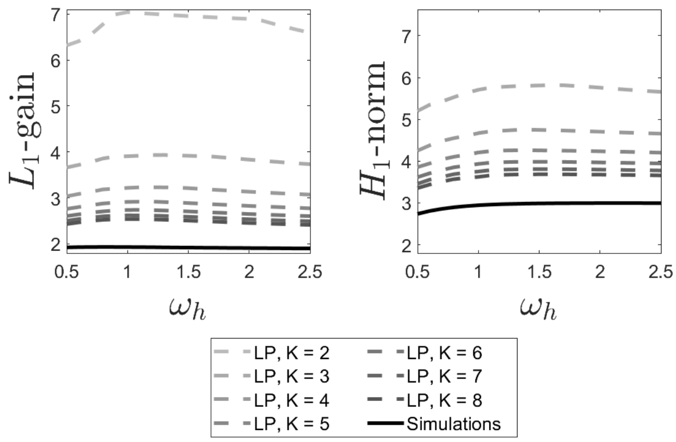

Fig. 2. Upper bound on L_1-gain and H_1-norm calculated using Theorem 7 for various triangulations $\mathcal{T}_K^F$ (grey). Additionally, using time-series simulations, a lower bound on the L_1-gain and an estimate of the H_1-norm are calculated (black).

5.2 Stability analysis

For multiple combinations (k_h, ω_h) we check whether the HIGS-controlled system is ISS using extensive time-series simulations (to serve as a baseline), the CPA method with Theorem 7 and the CPQ method with LMIs from van den Eijnden et al. (2022). The CPA method and the CPQ method will be given a similar amount of computation time ($\sim 10^3$ s), so that the comparison gives an indication of which method is computationally more efficient. See Fig. 1.

Notice that the CPA method is capable of using a finer triangulation ($\mathcal{T}_5^F$, which has $N = 1200$ simplicial cones) than the CPQ method ($\mathcal{T}_2^F$, which has $N = 192$ simplicial cones) given a similar amount of computation time. This is because the CPQ method involves a larger number of optimization variables given the same triangulation. Therefore, the CPQ method has a larger time complexity given the same triangulation. Despite this, however, it can be seen that the CPQ method can verify the stability of more combinations (k_h, ω_h) than the CPA method. This is due to the fact that the CPQ Lyapunov functions require a less dense triangulation than the CPA Lyapunov functions. When removing the restriction on computation time, it was found that the CPA method could find the same ISS combinations (k_h, ω_h) as the CPQ method. However, the CPA method required more computation time than the CPQ method to do so.

Note that in realistic industrial applications, the damping is typically much lower and the size of the HIGS-controlled system much larger than in the use case considered here. See for example the case in van den Eijnden et al. (2020). When studying such systems, it was found that the gap in accuracy between the CPA and CPQ methods only increased when the damping was lowered and/or when the size of the HIGS-controlled system increased.

5.3 Performance analysis

Next, we show how Theorem 7 can be used to calculate upper bounds on the L_1-gain and H_1-norm of the HIGS-controlled system. Note that with the CPQ method also upper bounds on these two metrics could be calculated, but it would require conservative estimates, e.g., related to the equivalence of norms. The value of k_h is fixed to 1 and the value of ω_h is varied over $[0.5, 2.5]$. Then, using Theorem 7, upper bounds on the L_1-gain and H_1-norm (corresponding to an initial condition $x(0) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$) are calculated. See Fig. 2, where for different triangulations $\mathcal{T}_K^F$ the L_1-gain and the H_1-norm are calculated using LP.

In Fig. 2, we also show the true value of the H_1-norm and a lower bound on the L_1-gain determined using time-series simulations. It can be seen that the LP problem defined in Theorem 7 produces an increasingly tight upper bound on the L_1-gain and H_1-norm as the refinement K of the triangulation $\mathcal{T}_K^F$ increases. The remaining difference between the calculated upper bounds and the results from the time-series simulations could be caused by conservatism in Theorem 7, numerical artifacts or limited computational power.

6. CONCLUSION

In this paper, extensions to the CPA method have been proposed to make it suitable for the ISS analysis of HIGS-controlled systems and conewise linear systems in general. A theorem was presented that poses conditions for the existence of a CPA Lyapunov function and upperbounds on the L_1-gain and H_1-norm of the system. These conditions can be checked with a linear programming problem that uses regional relaxations in its conditions. The proposed extensions to the CPA method were numerically demonstrated on a HIGS-controlled system. While it was shown that the modified CPA method can be used for the stability analysis of HIGS-controlled systems, it was also shown that this method appears to be computationally less efficient than the CPQ method. In order to make the CPA method a more reliable alternative, efforts will have to be made in order to improve the computational efficiency of this linear programming-based approach.

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