Application of the Schur Complement in Sum of Squares Optimisation

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Abstract: In this paper, we use the Schur Complement in combination with the sum of squares decomposition, first, to determine whether a nonlinear stochastic dynamical systems has a stable equilibrium and, second, to find a stabilising gain matrix for nonlinear dynamical systems. In both cases, we consider systems whose dynamics can be described using polynomial vector fields. Using many different examples, we highlight the effectivity of using our approaches. In some cases, we manage to obtain results that surpass previous ones. We believe that the presented approaches have many potential applications, for example, in the fields of aerospace and quantum control.

1 INTRODUCTION

The Schur complement, a fundamental concept in linear algebra and matrix theory, has found wide-ranging applications across various fields of science and engineering. Named after Issai Schur, who introduced it in the early 20th century, the Schur complement provides a powerful tool for analysing and simplifying complex matrix inequalities (Carlson et al., 1986). Pablo Parrilo connected the question whether a polynomial consists of a sum of squares (SOS) to modern optimisation via linear matrix inequalities and semidefinite programming (Parrilo, 2003). For dynamical systems consisting of polynomials of any degree, this strengthens the requirement of positivity to the condition that the polynomial function is a SOS. Crucially, to solve linear matrix inequalities, the results by Schur, are often applied (Boyd and Vandenberghe, 2004). The results by Schur are now a century old and those by Pablo Parillo just over two decades. Nevertheless, we continue to find novel applications of those to many problems in science and engineering. In this paper, we combine a novel use of the Schur complement and SOS decomposition methods to obtain global asymptotic stability (GAS) certificates for stochastic dynamical systems whose dynamics can be modelled through polynomial functions, and to design controllers guaranteeing GAS for polynomial nonlinear dynamical systems.

Natural as well as engineered systems have often complex dynamics that are affected by noise and disturbances. Additionally, they are difficult, if not impossible, to model perfectly, which leads to parameter uncertainty and approximations in the mathematical modelling. For biological systems, at times, noise carries information about the underlying network (Munsky et al., 2009; August, 2012). For this reason, one often requires models based on stochastic differential equations (SDE). Moreover, in many engineering fields, system modelling requires the use of nonlinear ordinary differential equations (Lavretsky and Wise, 2013; Slotine and Li, 1991), which often makes the design of a satisfactory feedback control law challenging (Iqbal et al., 2017).

1.1 Stochastic Differential Equations

One often requires models that are described by SDE, for path planning, modelling aerodynamic forces act-

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ing on aerial vehicles, or the design of ascent phase control for reusable launch vehicles that involves attitude manoeuvring through a wide range of flight conditions, including stochastic disturbances or non-deterministic disturbances, engine failure, and aero-surface locks (Berning Jr, 2020; Rodnishchev and Somov, 2018; Xu and Xin, 2011). Thus, modelling using SDE is important in aerospace science and engineering for the design of control strategies – for example, the interpolation between points of operation in gain scheduling can become otherwise prohibitively costly (to compute and store) – and path/mission planning, where it can provide safety certificates, to name but two.

Another example is finding feedback control laws that stabilise a pure state of a quantum system by means of continuous quantum non-demolition measurements. The preparation of such a state is crucial for quantum technologies. Quantum non-demolition measurements can be seen as “classical” ones (they do not prevent the wave function from collapsing). However, they do not destroy the particle under observation: In the double-slit experiment, information about a photon’s trajectory is gained without capturing/destroying the photon. Significantly, analysis and control of a stochastic dynamical system is a more challenging problem compared with its deterministic counterpart, particularly, for nonlinear stochastic dynamical systems (Ludyk, 2018; Cardona et al., 2018; Wiseman, 1994). Crucially, control strategies for quantum system pure state stabilisation are still in their infancy.

1.2 Nonlinear Control

Many effective techniques exist to design a controller for a nonlinear dynamical system. However, they typically ensure asymptotic stability only near the operating point of interest (local asymptotic stability). One common method is to linearise the set of differential equations either through Jacobian linearisation or feedback linearisation. Alternatively, one can try to solve the State-Dependent Riccati Equation, which is often very difficult, to derive an optimal control law (Cloutier, 1997). For systems described by polynomial vector fields, a powerful approach to controller design was developed in (August and Papachristodoulou, 2022). This approach, based on SOS decomposition, ensures asymptotic stability of the closed loop system.

1.3 Organisation of the Paper

The organisation of the paper is the following. In Section 2, we present the different methods used in this paper. More precisely, in Section 2.1, we present the Schur Complement and in Section 2.2 the SOS decomposition. In Section 2.3, we show how the two can be used to determine the stability of stochastic dynamical systems and similarly, in Section 2.4 how they can be used to design a stabilising controller for a nonlinear dynamical system, whose dynamics are described by means of polynomial functions. The main contributions of this paper are the convex problem relaxations presented in Sections 2.3.1 and 2.4.1 and their application in Section 3. For instance, we demonstrate the usefulness of the presented approaches in determining a stabilising controller for a quantum system as well as an aerospace system. Finally, we conclude the paper in Section 4.

2 METHODS

In this section, we present the different methods, including their novel application, used to obtain the results presented in Section 3.

2.1 Schur Complement

Thereom 1. *Consider,*

$$F = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix}.$$

If matrices A and C are symmetric then the following statements are equivalent:

i $F \succ 0$,
ii $A \succ 0$ *and* $C - B^{\mathrm{T}}A^{-1}B \succ 0$,
iii $C \succ 0$ *and* $A - BC^{-1}B^{\mathrm{T}} \succ 0$.

Similarly, the following statements are equivalent:

iv $F \prec 0$,
v $A \prec 0$ *and* $C - B^{\mathrm{T}}A^{-1}B \prec 0$,
vi $C \prec 0$ *and* $A - BC^{-1}B^{\mathrm{T}} \prec 0$.

This theorem can be traced back to Schur’s orginal paper (Schur, 1917) and similar results for semidefinite F exist. To see that $(i) \Leftrightarrow (ii)$, note that if M is nonsingular then $M^T FM \succ 0 \Leftrightarrow F \succ 0$ and that

$$\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C - B^{\mathrm{T}}A^{-1}B \end{bmatrix}.$$

All other equivalencies can be shown similarly. Matrices $C - B^{\mathrm{T}}A^{-1}B$ and $A - BC^{-1}B^{\mathrm{T}}$ are called the *Schur Complements*.

2.2 Sum of Squares Decomposition

Testing for non-negativity of real-valued polynomial function $F(x)$ of degree $2d$ is NP-hard (Murty and Kabadi, 1987), $x \in \mathbb{R}^n$. However, a sufficient condition for $F(x)$ to be nonnegative is that it can be decomposed into SOS (Parrilo, 2003):

$$F(x) = \sum_i f_i^2(x) \geq 0,$$

where f_i are polynomial functions. Now, $F(x)$ is SOS if and only if there exists a matrix R such that

$$F(x) = \sum_i f_i^2(x) = \chi^{\mathrm{T}} R \chi,\ R = R^{\mathrm{T}} \succeq 0, \tag{1}$$

$$\chi^{\mathrm{T}} = \begin{bmatrix} 1 & x_{(1)} & \dots & x_{(n)} & x_{(1)}x_{(2)} & \dots & x_{(n)}^d \end{bmatrix}.$$

The entries of vector χ consist of all monomial combinations of the elements of vector x up to degree d (including $x_{(i)}^0 = 1$) and, thus, its length is $\ell = \binom{n+d}{d}$. While R is usually non-unique, (1) poses certain constraints on it of the form

$$\mathrm{tr}(A_j R) = c_j, \quad j = 1, 2, \dots, m,$$

where $\mathrm{tr}(M)$ denotes the trace of square matrix M and matrices A_j and constants c_j are problem dependent.

As illustration consider

$$\begin{aligned} F(x) &= 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4 \\ &= \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix} \\ &= q_{11}x_1^4 + q_{22}x_2^4 + (q_{33} + 2q_{12})x_1^2x_2^2 \\ &\quad + 2q_{13}x_1^3x_2 + 2q_{23}x_1x_2^3. \end{aligned}$$

For $\chi_1 = x_1^2$, $\chi_2 = x_2^2$, and $\chi_3 = x_1x_2$, we obtain the following linear equalities:

$$q_{11} = 2,\ q_{22} = 5,\ q_{33} + 2q_{12} = -1,\ 2q_{13} = 2,\ 2q_{23} = 0.$$

Thus, $j = 1, \dots, 5$, and, for example, $c_1 = 2$, $c_3 = -1$,

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In order to find R, we solve the optimisation problem associated with the following semidefinite programme, where matrix A_0 can be chosen to select a particular solution R:

$$\begin{aligned} \min \quad & \mathrm{tr}(A_0 R) \\ \text{s.t.} \quad & \mathrm{tr}(A_j R) = c_j,\ j = 1, \dots, m \\ & R = R^{\mathrm{T}} \succeq 0. \end{aligned} \tag{2}$$

In this paper, to solve (2), we use `SOSTOOLS` (Papachristodoulou et al., 2021) and the SeDuMi solver (Sturm, 1999).

2.3 Stochastic Differential Equations

Consider the following SDE,

$$\mathrm{d}\mathbf{X} = \mathbf{b}(\mathbf{X})\mathrm{d}t + \sigma(\mathbf{X})\mathrm{d}\mathbf{W}, \tag{3}$$

where $\mathbf{X}$ is the random state vector, $\mathbf{W}$ is an m-dimensional Wiener Process, $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ the drift function and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ the state-dependent diffusion matrix (Ventsel and Freidlin, 1970; Gihman and Skorohod, 1972).[1] Moreover, we assume that both maps, $\mathbf{b}$ and σ, satisfy the usual conditions for the existence and uniqueness of solutions (El-Samad and Khammash, 2004). For the reminder of this paper, we use the short-form notation σ for $\sigma(\mathbf{X})$. Let the origin be an equilibrium point of system (3). The associated infinitesimal generator is given by

$$AV(\mathbf{X}) = \sum_{i=1}^{n} \left[\mathbf{b}_i(\mathbf{X}) \frac{\partial V(\mathbf{X})}{\partial \mathbf{X}_i} + \frac{1}{2} \sum_{j=1}^{n} (\sigma\sigma^{\mathrm{T}})_{i,j} \frac{\partial^2 V(\mathbf{X})}{\partial \mathbf{X}_i \mathbf{X}_j} \right]$$

for any given function $V \in C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{\mathbf{0}\})$. If $V(\mathbf{0}) = 0$, $AV(\mathbf{0}) = 0$, and for all $\mathbf{X} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $V(\mathbf{X}) > 0$ and $AV(\mathbf{X}) < 0$ then the expected value of $V(\mathbf{X})$ decreases with time, where $\mathbf{X}$ evolves according to (3) (Øksendal, 2003).

Now, let

$$\mathrm{d}\mathbf{X} = A(\mathbf{X})\mathbf{X}\mathrm{d}t + G(\mathbf{X})\mathbf{X}\mathrm{d}\mathbf{W}, \tag{4}$$

where $\mathrm{d}\mathbf{W}$ is 1-dimensional and $G(x)$ and $A(x) \in \mathbb{R}[x]^{n\times n}$ are state-dependent matrices, and positive definite matrix Q be such that

$$V(\mathbf{X}) = \left(\mathbf{X}^{\mathrm{T}} Q \mathbf{X}\right)^{\frac{p}{2}},\ p > 0.$$

Then, in the following, we provide conditions for the associated $AV(\mathbf{X})$ to be negative definite (Hafstein et al., 2018), where for clarity we use x instead of $\mathbf{X}$. First note that

$$\begin{aligned} \mathbf{b}_i(x) &= \sum_{j=1}^{n} A_{i,j}(x) x_j, \\ V(x) &= \left(\sum_{i=1}^{n} x_i \sum_{j=1}^{n} Q_{i,j} x_j \right)^{\frac{p}{2}}, \\ \frac{\partial V(x)}{\partial x_i} &= \frac{p}{2} \left(x^{\mathrm{T}} Q x\right)^{\frac{p-2}{2}} 2 \sum_{j=1}^{n} Q_{i,j} x_j. \end{aligned} \tag{5}$$

[1] When noise in (3) is intrinsic, is it also termed vanishing stochastic perturbations, since then $\sigma(\mathbf{0}) = \mathbf{0}$.

Then, it follows from (5) that

$$\sum_{i=1}^{n} \mathbf{b}_i(x)\frac{\partial V(x)}{\partial x_i}$$
$$= p\left(x^{\mathrm{T}}Qx\right)^{\frac{p-2}{2}} \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} x_k Q_{k,i} A_{i,j}(x) x_j$$
$$= p\left(x^{\mathrm{T}}Qx\right)^{\frac{p-2}{2}} x^{\mathrm{T}}QA(x)x$$
$$= p\left(x^{\mathrm{T}}Qx\right)^{\frac{p-4}{2}} x^{\mathrm{T}}Qxx^{\mathrm{T}}QA(x)x$$
$$= -\frac{1}{2}p\left(x^{\mathrm{T}}Qx\right)^{\frac{p-4}{2}} H_0(x).$$

Now, for $q = 2-p$,

$$\frac{\partial^2 V(x)}{\partial x_i x_j} = p\frac{\partial}{\partial x_j}\left(x^{\mathrm{T}}Qx\right)^{\frac{p-2}{2}} \sum_{k=1}^{n} Q_{i,k}x_k$$
$$= -qp\left(x^{\mathrm{T}}Qx\right)^{\frac{p-4}{2}} \sum_{k=1}^{n} Q_{i,k}x_k \sum_{\ell=1}^{n} Q_{j,\ell}x_\ell$$
$$+ px^{\mathrm{T}}Qx\left(x^{\mathrm{T}}Qx\right)^{\frac{p-4}{2}} Q_{i,j}.$$

Next, consider, with $y := G(x)x$,

$$\left(G(x)xx^{\mathrm{T}}G(x)^{\mathrm{T}}\right)_{i,j} = \left(yy^{\mathrm{T}}\right)_{i,j} = y_i y_j.$$

Then,

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\sigma\sigma^{\mathrm{T}})_{i,j}\frac{\partial^2 V(x)}{\partial x_i x_j}$$
$$= -\frac{1}{2}p\left(x^{\mathrm{T}}Qx\right)^{\frac{p-4}{2}}(H_1(x)+H_2(x)),$$

where

$$H_1(x) = q\sum_{k=1}^{n}\sum_{i=1}^{n} x_k Q_{k,i} y_i \sum_{\ell=1}^{n}\sum_{j=1}^{n} x_\ell Q_{\ell,i} y_j$$
$$= q\left(x^{\mathrm{T}}QG(x)x\right)^2$$

and

$$H_2(x) = -x^{\mathrm{T}}Qx\sum_{i=1}^{n}\sum_{j=1}^{n} y_i Q_{i,j} y_j = -x^{\mathrm{T}}Qxy^{\mathrm{T}}Qy.$$

Thus, $AV(x) < 0$ if $H(x) > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$, where

$$\begin{aligned} H(x) &= H_0(x) + H_1(x) + H_2(x) \\ &= -x^{\mathrm{T}}\left(2QA(x) + G(x)^{\mathrm{T}}QG(x)\right)xx^{\mathrm{T}}Qx \\ &\quad + q\left(x^{\mathrm{T}}QG(x)x\right)^2. \end{aligned} \tag{6}$$

2.3.1 Problem Relaxation

Note that, given $c \in \mathbb{R}$, for unknown Q, unless $p = 2$, $H(x) \geq c\left(x^{\mathrm{T}}x\right)^2$ is a bilinear matrix inequality, which is a non-convex problem. To make the problem convex, we first require that $Q \preceq \bar{c}I_n$, $\bar{c} \in \mathbb{R}$, and then make use of the following inequality

$$\frac{\left(x^{\mathrm{T}}QG(x)x\right)^2}{x^{\mathrm{T}}Qx} \geq \frac{\left(x^{\mathrm{T}}QG(x)x\right)^2}{\bar{c}},$$

which implies that $H(x) \geq$

$$-x^{\mathrm{T}}\left(2QA(x)+G(x)^{\mathrm{T}}QG(x)\right)xx^{\mathrm{T}}x\bar{c}+q\left(x^{\mathrm{T}}QG(x)x\right)^2.$$

Next, we replace $\left(x^{\mathrm{T}}QG(x)x\right)^2$ by $z^{\mathrm{T}}Xz$, where z contains only those monomials that are necessary for equality

$$z^{\mathrm{T}}Xz = \left(x^{\mathrm{T}}QG(x)x\right)^2, \tag{7}$$

to hold (if such a matrix X exists), and require that $z^{\mathrm{T}}Xz \geq \left(x^{\mathrm{T}}QG(x)x\right)^2$, which can be written as convex constraint using the Schur Complement (see (8)). Thus, if state-dependent matrix $G(x)$ has only polynomial entries then to find a feasible solution of (6) such that $H(x) \geq c\left(x^{\mathrm{T}}x\right)^2$, we solve the following SOS optimisation problem, where we seek to enforce (7) by minimising the trace of matrix X:

$$\begin{array}{ll} \text{given} & A(x), G(x) \in \mathbb{R}[x]^{n\times n},\ \bar{c}\in\mathbb{R},\ c\in\mathbb{R},\ q \\ \text{min.} & \operatorname{tr}(X) \\ \text{s. t.} & \bar{H}(x) - c\left(x^{\mathrm{T}}x\right)^2 \text{ is SOS } \forall x\in\mathbb{R}^n \\ & \bar{H}(x) = -x^{\mathrm{T}}(2QA(x)+G(x)^{\mathrm{T}}QG(x))xx^{\mathrm{T}}x\bar{c} \\ & \qquad +qz^{\mathrm{T}}Xz \\ & Q\succeq I_n,\ Q\preceq \bar{c}I_n,\ X\succeq 0 \\ & y^{\mathrm{T}}M(x)y \text{ is SOS } \forall x\in\mathbb{R}^n \ \&\ \forall y\in\mathbb{R}^2 \\ & M(x) = \begin{bmatrix} z^{\mathrm{T}}Xz & x^{\mathrm{T}}QG(x)x \\ x^{\mathrm{T}}QG(x)x & 1 \end{bmatrix}. \end{array} \tag{8}$$

If the solution of (8) is such that (7) holds for matrix X then $H(x) \geq \bar{H}(x)$ and the expected value of (4) decreases with time.

Remark 1. *Note that if z contains not only those monomials that are necessary for equality $z^{\mathrm{T}}Xz = \left(x^{\mathrm{T}}QG(x)x\right)^2$ to hold then there might exist a matrix $\tilde{X}$, for which $\operatorname{tr}(\tilde{X}) < \operatorname{tr}(X)$ holds, while $z^{\mathrm{T}}\tilde{X}z > z^{\mathrm{T}}Xz = \left(x^{\mathrm{T}}QG(x)x\right)^2$ also holds, as the following example shows.*

For instance, consider

$$Q = \begin{bmatrix} 1.300976011928209 & 0.176426076324787 \\ 0.176426076324787 & 1.103417407153368 \end{bmatrix}$$

and

$$G = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}.$$

Then, $z^{\mathrm{T}}Xz = (x^{\mathrm{T}}QGx)^2$, where $z^{\mathrm{T}} = \begin{bmatrix} x_1^2 & x_1x_2 \end{bmatrix}$,

$$X = \begin{bmatrix} 0.124504641629441 & 0.778686414770154 \\ 0.778686414770154 & 4.870119897636248 \end{bmatrix}$$

and $\mathrm{tr}(X) = 4.9946$*. However, if we allow "unnecessary" monomials in z to appear – that is, those that would appear if matrix G were non-singular – then* $z^{\mathrm{T}} = \left[\begin{array}{ccc} x_1^2 & x_1x_2 & x_2^2 \end{array}\right]$*, and, for example, for* $\tilde{X} =$

$$\left[\begin{array}{ccc} 0.005258248861173 & 0.023272688728787 & 0.154806237483972 \\ 0.023272688728787 & 0.110381268124464 & 0.732647367566061 \\ 0.154806237483972 & 0.732647367566061 & 4.878012769236294 \end{array}\right],$$

where $\mathrm{tr}(\tilde{X}) = 4.99365$*, the following holds,*

$$z^T\tilde{X}z > (x^TQGx)^2.$$

2.4 Nonlinear Control

In this paper, we represent nonlinear dynamical systems, whose dynamics are governed by polynomial function, using the following notation:

$$\begin{aligned} &\dot{x} = A(x)x + Bu, \\ &x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m, \\ &A(x) \in \mathbb{R}[x]^{n\times n},\ B \in \mathbb{R}^{n\times m}, \end{aligned}$$

where x is the system state, u the input, $A(x)$ the state-dependent polynomial system matrix representing the system dynamics, B is the input matrix, and we denote the set of all polynomials by $\mathbb{R}[x]$. Particularly, $A(x) \in \mathbb{R}[x]^{n\times n}$ means that the following holds for its (i,j)-th entry, $A_{(i,j)} \in \mathbb{R}[x]$, where $i,j \in \{1,2,\ldots,n\}$. Although matrix B can also be state-dependent, for simplicity, we consider constant input matrices only. Now, consider

$$\begin{aligned} \dot{x} &= A(x)x + BK(x,k)x, \\ K(x,k) &= -B^{\mathrm{T}}Qf(x,k), \\ f(x,k) &= \sum_{i=0}^{k}(x^{\mathrm{T}}x)^{2k}. \end{aligned} \tag{9}$$

If the origin is an asymptotically stable equilibrium point of (9) then we seek a positive definite matrix Q such that we can prove this by means of the Lyapunov function given by $V(x) = x^{\mathrm{T}}Qx$, which implies that the following inequality must hold for all x, $x \neq 0$,

$$x^{\mathrm{T}}\left(QA(x)x - QBB^{\mathrm{T}}Qf(x,k)\right)x < 0. \tag{10}$$

2.4.1 Problem Relaxation

Note that (10) is a bilinear matrix inequality and, thus, a non-convex problem. To make this problem convex, we replace matrix $QBB^{\mathrm{T}}Q$ by matrix X and require that $X \succeq QBB^{\mathrm{T}}Q$, which can be written as convex constraint using the Schur Complement. Then, we try to obtain positive definite matrix Q by solving the following SOS optimisation problem, where we seek to enforce $X = QBB^{\mathrm{T}}Q$ by minimising the trace of matrix X:

$$\begin{array}{ll} \text{given} & A(x) \in \mathbb{R}[x]^{n\times n},\ B \in \mathbb{R}^{m\times n},\ c \in \mathbb{R},\ k \\ \text{minimise} & \mathrm{tr}(X) \\ \text{subject to} & x^{\mathrm{T}}(Xf(x,k) - QA(x))x \text{ is SOS } \forall x \in \mathbb{R}^n \\ & \left[\begin{array}{cc} X & QB \\ B^{\mathrm{T}}Q & I_m \end{array}\right] \succeq 0,\ Q \succeq cI_n \end{array} \tag{11}$$

If (11) has a feasible solution given by Q and $X = QBB^{\mathrm{T}}Q$ holds then matrix Q stabilises system (9).

3 RESULTS

In the following, we demonstrate the use of the approaches presented in this paper by applying them to different exemplary systems taken from the literature (Hafstein et al., 2018; Cardona et al., 2018; August and Papachristodoulou, 2022).

3.1 Linear SDE

3.1.1 Example 1: 2D System

Consider

$$A = \left[\begin{array}{cc} 0 & 2.5 \\ -2.5 & -0.9 \end{array}\right],\ G = \left[\begin{array}{cc} 0 & 0 \\ 0 & -2 \end{array}\right].$$

First, note that, for this system, the method presented in (Hafstein et al., 2018) was unable to find a matrix Q, for which $H(x) > 0$ holds. Now, for $p = 0.1$, $c = 0.3$, and $\bar{c} = 1.41$, we solve (8) and determine that $H(x) \geq c\left(x^{\mathrm{T}}x\right)^2$ for

$$Q = \left[\begin{array}{cc} 1.3010 & 0.1764 \\ 0.1764 & 1.1034 \end{array}\right]$$

by means of the following `SOSTOOLS` programme:

```
A=[0 2.5;-2.5 -.9];
G=[0 0;0 -2];
p=0.1;
q=2-p;
c=0.3;
cb=1.41;
solver_opt.solver='sedumi';
pvar x1 x2 v1 v2
x=[x1 x2]';
v=[v1 v2]';
p=sosprogram([x;v]);
[p,Q]=sospolymatrixvar(p,monomials(
    x,0),[2 2],'symmetric');
[p,X]=sospolymatrixvar(p,monomials(
    x,0),[2 2],'symmetric');
```

```
[~,z,~]=findsos((x'*rand(2)*G*x)^2)
p=sosineq(p,x'*(Q-eye(2))*x);
p=sosineq(p,x'*(cb*eye(2)-Q)*x);
H=-x'*(2*Q*A+G'*Q*G)*x*x'*x*cb...
    +q*z'*X*z;
p=sosineq(p,H-c*(x'*x)^2);
p=sosineq(p,x'*X*x);
p=sosineq(p,v'*[z'*X*z x'*Q*G*x;...
    x'*Q*G*x 1]*v);
p=sossetobj(p,trace(X));
[p,~]=sossolve(p,solver_opt);
Q=double(sosgetsol(p,Q))
sosgetsol(p,z'*X*z)-(x'*Q*G*x)^2
```

3.1.2 Example 2: 3D System

Consider

$$A=\begin{bmatrix} 0.2759 & 0.0831 & -0.9603 \\ 0.8794 & -0.8281 & 0.2348 \\ -0.3879 & -1.8186 & -0.1508 \end{bmatrix},$$

and

$$G=\begin{bmatrix} 3.3849 & -0.3554 & 0.1794 \\ -0.3554 & 2.2541 & 0.3234 \\ 0.1794 & 0.3234 & 2.8609 \end{bmatrix}.$$

For $p=0.1$, $c=0.1$, and $\bar{c}=1$, using a `SOSTOOLS` programme similar to the one used in Example 3.1.1 to solve (8), we determine that $H(x)\geq c\left(x^{\mathrm{T}}x\right)^2$ for $Q=I$.

3.2 Density Matrix of a Two-State Quantum System

A *two-state system* is a quantum system that consists of two independent – that is, physically distinguishable – quantum states and all their quantum superpositions. For example, the Stern-Gerlach experiment is such a system, where the distribution of magnetic moments is not a continuous one but is limited to two values. Moreover, there are matrices, so-called *observables*, associated with such a system. If A is an observable then the measurement of a physical entity corresponding to observable A must be an eigenvalue λ_i of A (Ludyk, 2018). Now, for the eigenvalues to be always real, observable A must be Hermitian and, it follows from the above, that the state variable can be described by a linear combination of its eigenvectors.

The density matrix of a two-state quantum system describes an ensemble of states. Let it be given by density operator

$$\rho=\frac{1}{2}\left(I+\sigma_x x+\sigma_y y+\sigma_z z\right)=\frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix},$$

where σ_x, σ_y, and σ_x are the Pauli matrices given by

$$\sigma_x=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma_y=\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \sigma_z=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and real scalars x, y, and z satisfy

$$x^2+y^2+z^2\leq 1.$$

We would like to drive ρ towards a pure state, that is, towards $z=1$ or $z=-1$ by means of non-demolition measurement and using feedback; for example, by first entangling the system and then using the (possibly destructive) measurement of the entangled system for feedback. For instance, such feedback can be used for quantum-state preparation, which is crucial for quantum technologies.

The quantum closed system evolves as in the following:

$$\dot{\rho}=-i[H,\rho]=-i(H\rho-\rho H),$$

where H is a Hermitian matrix and $[H,\rho]=H\rho-\rho H$. The quantum open (measured) system with feedback evolves as in the following (Cardona et al., 2018; Wiseman, 1994):

$$\begin{aligned} d\rho &= -if[\sigma_y,\rho]dt+\mathbf{D}(\sigma_z-ik\sigma_y,\rho)dt \\ &\quad +\mathbf{M}(\sigma_z,\rho)dW-i\kappa[\sigma_y,\rho]dW, \end{aligned} \tag{12}$$

where κ and f are tuneable control parameters,

$$\mathrm{tr}(\rho)=1,\ \rho=\begin{bmatrix} \rho_1 & \rho_2 \\ \rho_2^* & \rho_3 \end{bmatrix},$$

$$\mathbf{D}(L,\rho)=L\rho L^\dagger-\frac{1}{2}L^\dagger L\rho-\frac{1}{2}\rho L^\dagger L,$$

and

$$\mathbf{M}(L,\rho)=L\rho+\rho L^\dagger-\mathrm{tr}((L+L^\dagger)\rho)\rho.$$

Now, consider the following change of coordinates,

$$x=\mathrm{tr}(\rho\sigma_x),\ z=\mathrm{tr}(\rho\sigma_z)\Rightarrow x=\rho_2+\rho_2^*,\ z=\rho_1-\rho_3.$$

Then,

$$dx=2(2\kappa-x\kappa^2-x+fz)dt+2(\kappa-x)zdW$$

and

$$dz=-2(z\kappa^2+fx)dt+2(1-z^2-\kappa x)dW.$$

Next, we use also the following coordinate transformation: $\bar{z}=z-1$. For control parameters $\kappa=-\bar{z}$ and $f=(\bar{z}+2)\bar{z}$, as suggested in (Cardona et al., 2018), we obtain,

$$\begin{aligned} dx &= -2\left(2\bar{z}+x\bar{z}^2+x-(\bar{z}+2)(\bar{z}+1)\bar{z}\right)dt \\ &\quad -2(\bar{z}+x)(\bar{z}+1)dW \\ &= -2\left(x\bar{z}^2+x-\bar{z}^3-3\bar{z}^2\right)dt \\ &\quad -2\begin{bmatrix} \bar{z}+1 & \bar{z}+1 \end{bmatrix}\chi dW \\ &= -2\begin{bmatrix} \bar{z}^2+1 & -\bar{z}^2-3\bar{z} \end{bmatrix}\chi dt \\ &\quad -2\begin{bmatrix} \bar{z}+1 & \bar{z}+1 \end{bmatrix}\chi dW \end{aligned} \tag{13}$$

and

$$\begin{aligned} dz &= -2\left(\bar{z}^3+\bar{z}^2+\bar{z}^2x+2\bar{z}x\right)dt \\ &\quad -2(\bar{z}^2+2\bar{z}-\bar{z}x)dW \\ &= -2\left[\begin{array}{cc} \bar{z}^2+2\bar{z} & \bar{z}^2+\bar{z} \end{array}\right]\chi dt \\ &\quad -2\left[\begin{array}{cc} -\bar{z} & \bar{z}+2 \end{array}\right]\chi dW, \end{aligned} \tag{14}$$

where $\chi = \left[\begin{array}{c} x \\ \bar{z} \end{array}\right]$. Thus,

$$A(\chi) = -2\left[\begin{array}{cc} \bar{z}^2+1 & -\bar{z}^2-3\bar{z} \\ \bar{z}^2+2\bar{z} & \bar{z}^2+\bar{z} \end{array}\right]$$

and

$$G(\chi) = -2\left[\begin{array}{cc} \bar{z}+1 & \bar{z}+1 \\ -\bar{z} & \bar{z}+2 \end{array}\right].$$

To check for exponential stability of (13)–(14), for $p = 0.1$, $c = 0.065$, and $\bar{c} = 1$, we use a SOSTOOLS programme similar to the one used in Example 3.1.1 to solve (8) and determine that $H \geq c\left(x^{\mathrm{T}}x\right)^2$ for $Q = I$. Note that, in this section, matrices $A(\chi)$ and $G(\chi)$ are state dependent, as opposed to constant matrices A and G in Section 3.1. Finally, exponential stability of (13)–(14) has been already shown in (Cardona et al., 2018) and we do not claim to improve on these results, we rather aim at exemplifying our approach.

3.3 Nonlinear Control

The system dynamics of the examples in this section are all given by $\dot{x} = A(x)x + BKx$.

3.3.1 Tunnel Diode Circuit

The dynamics of a tunnel diode circuit are defined by

$$\begin{aligned} A(x) &= \left[\begin{array}{cc} -0.5g(x_1) & 0.5 \\ -0.2 & -0.3 \end{array}\right],\ B = \left[\begin{array}{c} 0 \\ 0.2 \end{array}\right], \\ g(x_1) &= 17.76 - 103.79x_1 + 229.62x_1^2 \\ &\quad - 226.31x_1^3 + 83.72x_1^4. \end{aligned}$$

Here, x_1 is the voltage across the capacitor and x_2 the current through the inductor. For $c = 1$ and $k = 0$, we solve (11) and obtain stabilising gain matrix $K = -\left[\begin{array}{cc} 0 & 0.2281 \end{array}\right]$ through the following SOSTOOLS programme (that also confirms that $X - QBB^{\mathrm{T}}Q \approx 0$):

```
solver_opt.solver='sedumi';
pvar x1 x2 v
x=[x1;x2];
w=[x;v];
p=sosprogram(w);
g=17.76-103.79*x1+229.62*x1^2 ...
    -226.31*x1^3+83.72*x1^4;
A=[-0.5*g 0.5; -0.2 -0.3];
B=[0;0.2];
[p,Q]=sospolymatrixvar(p,monomials(
    x,0),[2 2],'symmetric');
[p,X]=sospolymatrixvar(p,monomials(
    x,0),[2 2],'symmetric');
p=sosineq(p,x'*(Q-eye(2))*x);
p=sosineq(p,x'*(X-Q*A)*x);
p=sosineq(p,w'*[X Q*B;B'*Q 1]*w);
p=sossetobj(p,trace(X));
[p,~]=sossolve(p,solver_opt);
X=sosgetsol(p,X);
Q=sosgetsol(p,Q);
X-Q*B*B'*Q
K=-B'*Q
```

3.3.2 Air-Breathing Hypersonic Flight Vehicle

The longitudinal dynamics of a simplified version of an air-breathing hypersonic flight vehicle are defined by $A_{11} = 0$,

$$\begin{aligned} &A_{12} = -(0.645x_2 + 0.01921)(0.1247x_1 + 0.7370)^2, \\ &A_{21} = -0.0002706,\ A_{22} = -0.0574 - 0.009716x_1, \\ &B = \left[\begin{array}{c} 0.014 \\ 0 \end{array}\right]. \end{aligned}$$

Here, x_1 is the velocity, x_2 the angle of attack, and we use the fact that for small angles $\sin(x_2) \approx x_2$. Furthermore, u is the thrust input. For $c = 1$ and $k = 2$, using a SOSTOOLS programme similar to the one used in Example 3.3.1 to solve (11), we obtain stabilising gain matrix $K = -\left[\begin{array}{cc} 0.0140 & 0 \end{array}\right]\left(1 + x^{\mathrm{T}}x + \left(x^{\mathrm{T}}x\right)^2\right)$.

3.3.3 Lorenz System

Consider the Lorenz system defined by

$$A(x) = \left[\begin{array}{ccc} 10 & -10 & 0 \\ 28 & -1 & -x_{(1)} \\ x_{(2)} & 0 & 8 \end{array}\right],\ B = \left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{array}\right].$$

For $c = 11$ and $k = 0$, using a SOSTOOLS programme similar to the one used in Example 3.3.1 to solve (11), we obtain stabilising gain matrix $K = -11B^{\mathrm{T}}$.

4 CONCLUSIONS

In this paper, we presented two related approaches to either determine whether a nonlinear stochastic dynamical system has a stable equilibrium or to find a stabilising gain matrix for a nonlinear dynamical system, where we considered systems whose dynamics can be described using polynomial vector fields. Significantly, by using the Schur Complement in combination with the sum of squares decomposition, we provided convex alternatives to bilinear matrix inequalities. Using different examples, we highlighted the effectivity of using our approaches, which also managed to obtain results that surpassed previous ones. We believe that the presented approaches have many potential applications, for example, in the fields of aerospace and quantum control.

REFERENCES

August, E. (2012). Using noise for model-testing. *Journal of Computational Biology*, 19:968–977.

August, E. and Papachristodoulou, A. (2022). Feedback control design using sum of squares optimisation. *European Journal of Control*, page 100683.

Berning Jr, A. (2020). *Control and Optimization for Aerospace Systems with Stochastic Disturbances, Uncertainties, and Constraints*. PhD thesis, University of Michigan.

Boyd, S. and Vandenberghe, L. (2004). *Convex Optimization*. Cambridge University Press, Cambridge, UK.

Cardona, G., Sarlette, A., and Rouchon, P. (2018). Exponential stochastic stabilization of a two-level quantum system via strict lyapunov control. In *2018 IEEE Conference on Decision and Control*, pages 6591–6596.

Carlson, D., Markham, T. L., and Uhlig, F. (1986). Emilie haynsworth, 1916-1985. *Linear Algebra and its Applications*, 75:269–276.

Cloutier, J. (1997). State-dependent riccati equation techniques: an overview. In *Proceedings of the 1997 American Control Conference (Cat. No.97CH36041)*, pages 932–936.

El-Samad, H. and Khammash, M. (2004). Stochastic stability and its application to the analysis of gene regulatory networks. *Proceedings of the American Control Conference*, pages 3001–3006.

Gihman, I. I. and Skorohod, A. V. (1972). *Stochastic Differential Equations*. Springer-Verlag, Berlin.

Hafstein, S., Gudmundsson, S., Giesl, P., and Scalas, E. (2018). Lyapunov function computation for autonomous linear stochastic differential equations using sum-of-squares programming. *Discrete and Continuous Dynamical Systems - B*, 23(2):939–956.

Iqbal, J., Ullah, M., Khan, S. G., Khelifa, B., and Ćuković, S. (2017). Nonlinear control systems - a brief overview of historical and recent advances. *Nonlinear Engineering*, 6(4):301–312.

Lavretsky, E. and Wise, K. (2013). *Robust and Adaptive Control: With Aerospace Applications*. Springer-Verlag London, London, United Kingdom.

Ludyk, G. (2018). *Quantum Mechanics in Matrix Form*. Springer, Cham, Switzerland.

Munsky, B., Trinh, B., and Khammash, M. (2009). Listening to the noise: random fluctuations reveal gene network parameters. *Molecular Systems Biology*, 5.

Murty, K. G. and Kabadi, S. N. (1987). Some NP-complete problems in quadratic and nonlinear programming. *Math. Program.*, 39:117–129.

Øksendal, B. (2003). *Stochastic Differential Equation: An Introduction with Applications*. Springer-Verlag, Berlin, 6 edition.

Papachristodoulou, A., Anderson, J., Valmorbida, G., Prajna, S., Seiler, P., Parrilo, P. A., Peet, M. M., and Jagt, D. (2021). *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*. http://arxiv.org/abs/1310.4716. https://github.com/oxfordcontrol/SOSTOOLS.

Parrilo, P. A. (2003). Semidefinite programming relaxations for semialgebraic problems. *Math. Program., Ser. B*, 96:293–320.

Rodnishchev, N. and Somov, Y. (2018). Control optimization in aerospace engineering at stochastic perturbations and stream of faults. In *2018 5th IEEE International Workshop on Metrology for AeroSpace (MetroAeroSpace)*, pages 171–175.

Schur, I. (1917). Über Potenzreihen, die im Inneren des Einheitskreises beschränkt sind. *J. Reine Angew. Math.*, 147:205–232.

Slotine, J. J. E. and Li, W. (1991). *Applied Nonlinear Control*. Prentice Hall, Englewood Cliffs, NJ, USA.

Sturm, J. F. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12:625–653. Available at http://sedumi.ie.lehigh.edu.

Ventsel, A. D. and Freidlin, M. I. (1970). On Small Random Perturbations of Dynamical Systems. *Russ. Math. Surv.*, 25:1.

Wiseman, H. M. (1994). Quantum theory of continuous feedback. *Phys. Rev. A*, 49:2133–2150.

Xu, Y. and Xin, M. (2011). Nonlinear stochastic control for space launch vehicles. *IEEE Transactions on Aerospace and Electronic Systems*, 47(1):98–108.