Numerical computations of Lyapunov functions for switched linear systems*

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Abstract. We compare different optimisation methods for the computation of Lyapunov functions for switched linear systems. In particular, we compare methods based on linear programming to methods based on semi-definite optimisation. For the linear programming methods, we describe two approaches aimed at reducing the size of the linear programme. This is done by using different kinds of coordinate transforms. One attempts to increase the rotational symmetry of the solution trajectories of the sub-systems. The other one rotates the areas of maximum curvature of possible Lyapunov functions to where the triangulation is closer meshed. We investigate the effect of these coordinate transforms on several examples and compare their efficiency.

Keywords: Switched Systems, Lyapunov Functions, Semidefinite Programming, Linear Programming, Preconditioning.

1 Introduction

For the understanding of dynamical systems in science and engineering, one often uses Ordinary Differential Equations (ODEs) for modelling, that is, the temporal behaviour of the state variables is quantified through a differential equation of the following form,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

where $\mathbf{x}$ is the so called state vector and the vector valued function $\mathbf{f}$ describes the dynamics of the system.

Switched systems are used for modelling in many different fields [25]. They describe systems, whose dynamics are affected by instantaneous changes, by considering a set of continuous-time sub-systems and a rule governing the switching

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between them. Examples for their use range from the biological sciences [15], for example, the dynamics of human body thermoregulation during sleep abruptly changes when sleep transitions from non-rapid eye movement (REM) sleep to REM sleep [22], to mechanical engineering, where one example is the dynamics of an engine with shifting gears [40]. Other engineering applications are power electronics, automotive control, robotics, and air traffic control [44], to name a few. Analysing switched systems is also important in the field of hybrid systems [25], where the switching is due to the interaction between the dynamics of individual continuous-time sub-systems and the discrete-time dynamics of the switching [18]. The differential inclusions associated with uncertainty quantification in continuous-time systems is another common source of switched systems [5, 13].

A switched linear system has the following form,

$$\dot{\mathbf{x}} = A_m\mathbf{x}, \quad m \in \mathcal{P}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{2}$$

The switching between the systems $\dot{\mathbf{x}} = A_m\mathbf{x}$ is modelled through a switching signal $\sigma\colon [0,\infty) \to \mathcal{P}$. In this paper, we will only consider arbitrary switching between a finite number of sub-systems; that is, $\mathcal{P}$ is finite and $\sigma\colon [0,\infty) \to \mathcal{P}$ is arbitrary, except for the (technical) assumption that the number of switching times is finite on any finite time-interval.

Parts of the results in this paper were presented at the International Conference on Informatics in Control, Automation and Robotics (ICINCO) in Rom, Italy, 2023, and published in the corresponding conference proceedings [3]. It is organised as follows. In Section 2, we recall results on stability of equilibrium points, switched systems, linear programming (LP), and semidefinite programming (SDP). In sections 3 and 4, we discuss our LP approach and the canonical SDP approach, respectively, to compute Lyapunov functions for switched linear system. In Section 4, we compare the LP and the SDP approaches using numerous examples. In Section 5, we discuss the AngleAnalysis app, used to aid the LP approach, before we conclude the paper in Section 6.

2 Preliminaries

In this section, we recall some results and definitions used in this paper regarding stability of equilibrium points, switched system, LP, and SDP.

2.1 Stability of an Equilibrium Point

When modelling a dynamical system using (1), the trajectory, or solution curve, is given by $t \mapsto \boldsymbol{\phi}(t,\boldsymbol{\xi})$, where $\boldsymbol{\xi}$ is the initial state; that is, $\boldsymbol{\phi}(0,\boldsymbol{\xi}) = \boldsymbol{\xi}$. In the following, we provide the definitions of an equilibrium point for system (1) and of the stability of an equilibrium point. We denote the Euclidian norm of a vector $\mathbf{x} \in \mathbb{R}^n$ by $\|\mathbf{x}\|$.

Definition 1. *A point* $\mathbf{x}_0$ *is an* equilibrium point *for* (1) *if* $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$. *Without loss of generality, we assume that the equilibrium point of interest is located at* *the origin; otherwise, we can rewrite* (1) *in a new set of coordinates given by* $\mathbf{z} = \mathbf{x} - \mathbf{x}_0$.

Definition 2. *The origin is called* asymptotically stable *for* (1) *if*

(i) for every $\varepsilon > 0$ *there exists a* $\delta > 0$*, such that if* $\|\boldsymbol{\xi}\| < \delta$ *then for every* $t \geq 0$ *we have* $\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\| < \varepsilon$ *and*
(ii) there exists a $\delta > 0$*, such that if* $\|\boldsymbol{\xi}\| < \delta$ *then* $\lim_{t\to\infty} \|\boldsymbol{\phi}(t, \boldsymbol{\xi})\| = 0$.

The origin is called exponentially stable *if there exist constants* $\alpha > 0$ *and* $c \geq 1$ *such that*

$$\|\boldsymbol{\phi}(t, \boldsymbol{\xi})\| \leq c\|\boldsymbol{\xi}\| \exp[-t\alpha] \quad \text{for all } t \geq 0. \tag{3}$$

2.2 Switched Systems

When modelling switching behaviour by switching between distinct $\mathbf{f}_m(\mathbf{x})$, $m \in \mathcal{P}$ where $\mathcal{P}$ is some index set, such that

$$\dot{\mathbf{x}} = \mathbf{f}_{\sigma(t)}(\mathbf{x}), \quad \sigma : [0, \infty[\mapsto \mathcal{P}, \tag{4}$$

we call (4) a switched system with switching signal σ and sub-systems $\mathbf{f}_m$. On first sight, one might be tempted to derive the stability of the equilibrium for (4) by considering its stability for each sub-system. However, as Figure 1 shows, switching can either stabilise two previously unstable systems or destabilise two stable ones. Consequently, we need to consider the shape of the switching signal when evaluating the stability of an equilibrium point of (4). There are several possibilities to adapt the stability concepts from definitions 1 and 2 to switched system (4). However, since we assume that the switching signal is an arbitrary one, besides having a finite number of switching events on a finite time interval, we do not need to include it in the stability analysis. Under this assumption, a necessary condition for the asymptotic stability of an equilibrium point for (4) is that the sub-systems $\dot{\mathbf{x}} = \mathbf{f}_m(\mathbf{x})$ all have the same equilibrium point and that it is asymptotically stable for each one of them. That is, for $\mathbf{x}_0 = \mathbf{0}$, we demand $\mathbf{f}_m(\mathbf{0}) = \mathbf{0}$ for every $m \in \mathcal{P}$ and that the conditions of Definition 2 hold for all $\boldsymbol{\phi}(t, \boldsymbol{\xi}) = \boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi})$, where $t \mapsto \boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi})$ is the solution curve of (4) for a particular admissible switching signal σ.

Note that in science and engineering, one often linearises the system given by (1) and it becomes

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad A := D\mathbf{f}(\mathbf{0}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{5}$$

where $D\mathbf{f}(\mathbf{0}) \in \mathbb{R}^{n\times n}$ denotes the Jacobian-matrix of $\mathbf{f}$ at the origin. According to the Hartman-Grobman theorem [19], the solutions of linearised system (5) have the same qualitative behaviour close to the origin as system (1), whenever the real parts of the eigenvalues of $D\mathbf{f}(\mathbf{0})$ are nonzero. The solution to the linear system is given by the matrix exponential $\exp[At]$; that is, a solution starting at $\boldsymbol{\xi} \in \mathbb{R}^n$ at time zero will be at $\exp[At]\boldsymbol{\xi}$ at time t or $\boldsymbol{\phi}(t, \boldsymbol{\xi}) := \exp[At]\boldsymbol{\xi}$. The

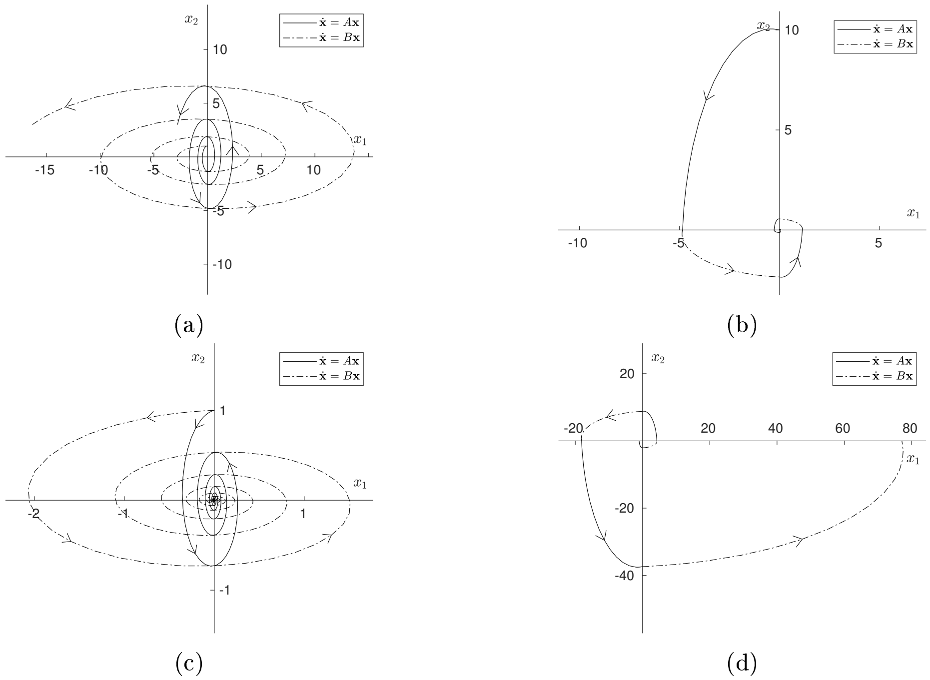


Fig. 1: Trajectories of (a) two unstable systems starting at $\mathbf{x} = [0, 1]^T$ with (b) a stable switching between those two systems starting $\mathbf{x} = [0, 10]^T$ and trajectories of (c) two stable systems starting at $\mathbf{x} = [0, 1]^T$ with (d) an unstable switching between those two systems starting $\mathbf{x} = [-1, 0]^T$. The figure is adapted from [2].

matrix valued function $t \mapsto \exp[At]$ is called the transition matrix of system (5). Note that asymptotic stability of the origin of linear system (5) is equivalent to its exponential stability. Moreover, the origin is asymptotically stable if and only if matrix A is Hurwitz; that is, the real part of all eigenvalues of A is negative.

Now, the transition matrix for (2) with switching signal σ is given by

$$t \mapsto \exp[A_{\sigma(t_k)}(t - t_k)] \cdot \exp[A_{\sigma(t_{k-1})}(t_k - t_{k-1})] \cdot \\ \cdots \exp[A_{\sigma(t_1)}(t_2 - t_1)] \cdot \exp[A_{\sigma(t_0)}(t_1 - t_0)],$$

where $t_0, t_1, t_2, \ldots, t_k$ are the switching times and

$$t > t_k > t_{k-1} > \ldots > t_1 > t_0 = 0.$$

Moreover, for switched system (2) under arbitrary switching, the origin is exponentially stable if and only if for every switching signal σ with a finite number of switching times on every finite interval, there exist constants $\alpha > 0$ and $c \geq 1$, independently of σ, such that inequality (3) holds for

$$\begin{aligned}\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi}) := &\exp[A_{\sigma(t_k)}(t-t_k)]\cdot\exp[A_{\sigma(t_{k-1})}(t_k-t_{k-1})]\\ &\cdots\exp[A_{\sigma(t_1)}(t_2-t_1)]\cdot\exp[A_{\sigma(t_0)}(t_1-t_0)]\boldsymbol{\xi}.\end{aligned}$$

Notably, even for planar systems ($n = 2$) and switching between only two different systems ($\mathcal{P} = \{1, 2\}$), unless matrices A_m have some very special structure, for instance, if they commute [1], determining stability is an open problem. To understand why, note that exponential stability of the origin is characterised by matrices A_1 and A_2 and the origin is exponentially stable if and only if all pairwise convex combinations of A_1, A_2, A_1^{-1}, and A_2^{-1} are Hurwitz, a property which is difficult to prove [14, 24].

A sufficient condition for exponential stability of the origin of (2) is the existence of a quadratic common Lyapunov function (QCLF). That is, there exists an $n \times n$ symmetric positive definite matrix P, denoted by $P \succ 0$, such that the following linear matrix inequality (LMI) holds [23, 9]:

$$PA_m + A_m^{\mathrm{T}}P \prec 0, \quad \forall m \in \mathcal{P}. \tag{6}$$

However, even for $n = |\mathcal{P}| = 2$ the origin of an arbitrarily switched system can be exponentially stable, while a QCLF does not exist for the system. For published work on the existence of an common Lyapunov function (CLF), see for example [30, 31, 17, 27, 28]. For a review on different existing methods to compute such a CLF, if it exists, see [16]. One method to compute a CLF is to use LP to parametrise piecewise linear Lyapunov functions [11, 12, 6–8, 45, 37, 38, 33, 34, 46, 4]. Another one is to use SDP to parametrise higher-order polynomial Lyapunov functions [39, 36]. In the next two subsection we give a concise description of LP and SDP as needed in this paper.

2.3 Linear Programming

In LP one seeks to solve the following minimisation problem:

$$\text{minimise} \quad \mathbf{c}^T\mathbf{x} \tag{7}$$

$$\text{subject to} \quad A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}. \tag{8}$$

Here $A \in \mathbb{R}^{n\times m}$, $\mathbf{c} \in \mathbb{R}^m$, and $\mathbf{b} \in \mathbb{R}^n$ are given and one wants to find a feasible vector $\mathbf{x} \in \mathbb{R}^m$ – that is, such that $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ holds, where the inequalities are to be understood component-wise – that minimises $\mathbf{c}^T\mathbf{x}$. Typically m is much larger than n. Note that there are many other equivalent formulations of an LP problem. There exist very efficient methods to solve LP problems and mature solvers are available. In this work, we use Gurobi [20], which is a commercial solver but can be used free of charge within academia.

2.4 Semidefinite Programming

In SDP, we replace the nonnegative orthant constraint of LP by the cone of positive semidefinite matrices and pose the following minimisation problem [43]:

$$\begin{aligned} \text{minimise} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & F(\mathbf{x}) \succeq 0, \text{ where} \\ & F(\mathbf{x}) = F_0 + \sum_{i=1}^{n} x_i F_i. \end{aligned} \tag{9}$$

Here, $\mathbf{x} \in \mathbb{R}^n$ is the free variable. The so-called problem data, which are given a priori, are vector $\mathbf{c} \in \mathbb{R}^n$ and symmetric matrices $F_j \in \mathbb{R}^{m \times m}$, $j = 0, 1, \ldots, n$. The *dual* problem associated with the semidefinite program given by (9) is [43]:

$$\begin{aligned} \text{maximise} \quad & -\text{trace}(F_0 Z) \\ \text{subject to} \quad & \text{trace}(F_i Z) = c_i, \ i = 1, 2, \ldots, n \\ & Z \succeq 0. \end{aligned} \tag{10}$$

Here, the free variable is the symmetric matrix $Z \in \mathbb{R}^{m \times m}$. Note that solutions of primal problem (9) provides upper bounds on solutions of the dual and vice versa. This is called *weak duality* and holds since

$$\begin{aligned} \mathbf{c}^T\mathbf{x} + \text{trace}(F_0 Z) &= \sum_{i=1}^{n} c_i x_i + \text{trace}(F_0 Z) = \sum_{i=1}^{n} \text{trace}(F_i Z) x_i + \text{trace}(F_0 Z) \\ &= \text{trace}\left(\left[\sum_{i=1}^{n} F_i x_i + F_0 \right] Z \right) \geq 0. \end{aligned}$$

The last inequality holds true because of self-duality of the positive semidefinite cone [35]. If the inequality holds strictly then we speak of *strong duality.*

Programmes of this type can be solved efficiently using interior-point methods. The interested reader is referred to reference [43] and the excellent textbook by the same authors [10]. Moreover, convexity of the set of symmetric positive semidefinite matrices in (9) implies that the minimisation problem has a global minimum, if it is feasible.

3 Lyapunov functions by Linear Programming

In the LP approach, presented in [4], for computing piecewise linear CLF for the switched linear system (2), a neighbourhood of the origin must first be triangulated. Then, an LP is created. Its variables are the values of the piecewise linear CLF to be parametrised for the system and its constraints secure that a feasible solution delivers a function that fulfils the conditions of a CLF. The number of the constraints of the LP problem is given by the number of simplices in the triangulation multiplied with the dimension of the state space. The

Table 1: Some examples of the number of simplices in the triangulation $\mathcal{T}_K$ in different dimensions n; see formula (11)

n	K	Number of simplices in $\mathcal{T}_K$
2	5	40
2	10	80
2	50	400
2	100	800
3	5	1,200
3	10	4,800
3	50	120,000
3	100	480,000
4	5	48,000
4	10	384,000
4	50	48,000,000
4	100	384,000,000
5	5	2,400,000
5	10	38,400,000
5	50	$2.5 \cdot 10^{10}$
5	100	$3.84 \cdot 10^{11}$

vertices of the simplices of the triangulation are first put on the integer grid $\mathbf{z} \in \mathbb{Z}^n$, $\|\mathbf{z}\|_\infty = K$, where $K \in \mathbb{N}$ determines the resolution, or fineness, of the triangulation and $\|\mathbf{z}\|_\infty := \max_{i=1,2,\ldots,n} |z_i|$. The triangulation with resolution parameter K is denoted $\mathcal{T}_K$. We will later map it to the triangulation $\mathcal{T}_K^{\mathbf{F}}$ for computational reasons, but the number of simplices will stay the same.

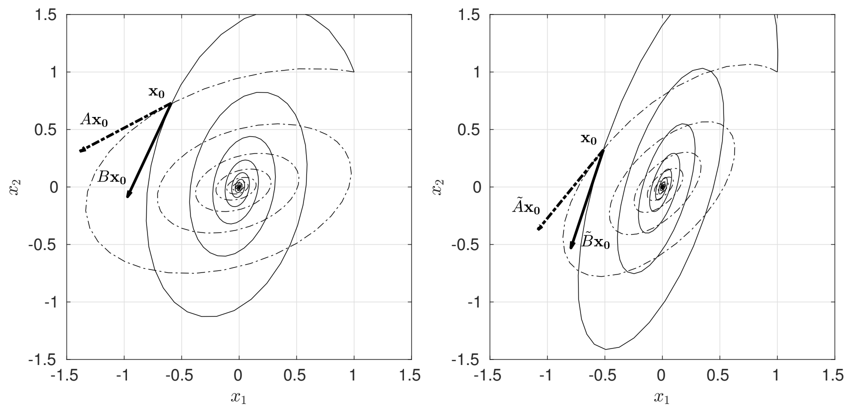
To count the total number of simplices in the triangulation $\mathcal{T}_K$ for a given dimension n, first, note that the triangulation subspace has $2n$ sides. Each side consists of $(2K)^{n-1}$ hypercubes of dimension $n-1$ and each hypercube leads to $(n-1)!$ simplices. To see this, note that the k-dimensional hypercube $[0,1]^k$ is cut into simplices according to $0 \leq x_{p(1)} \leq x_{p(2)} \leq \ldots \leq x_{p(k)} \leq 1$ for every permutation p of $\{1, 2, \ldots, k\}$ and that the $(n-1)$ dimensional hypercubes are cut accordingly. Thus, the number of simplices as a function of dimension n and resolution parameter K is given by: [3]

$$\text{Number of simplices} = 2n(2K)^{n-1}(n-1)! = 2^n K^{n-1} n! \tag{11}$$

We can see that this is a case of the curse of dimensionality, as the number of simplices grows very fast with dimension. For an example of the number of simplices needed, see Table 1.

In [21], it was shown that the numerical approach from [4] is always able to compute a CLF if one exists. However, as shown in Table 1, if the LP approach from [4] is to be applicable in dimensions larger than $n = 3$ or $n = 4$ then some reduction in the number of simplices is clearly needed. A promising preconditioning approach was presented in [4]. It attempts to increase the rotational symmetry of the solutions of the sub-systems (see Figure 2). In more

[3] Note that there is an error in [4] where $2n$ was erroneously replaced by 2^n



(a) Angle is 37.11°.

(b) Angle is now 20.97°.

Fig. 2: Exemplary angle between (a) vector fields $\mathbf{x} \mapsto A\mathbf{x}$ and $\mathbf{x} \mapsto B\mathbf{x}$ and (b) the transformed (or preconditioned) vector fields $\mathbf{x} \mapsto \tilde{A}\mathbf{x}$ and $\mathbf{x} \mapsto \tilde{B}\mathbf{x}$ at a point $\mathbf{x}_0$. The figure is adapted from [3].

detail, the idea is to make a coordinate transform such that $V(\mathbf{x}) = \|\mathbf{x}\|$ is closer to fulfilling the conditions of a CLF. To compute such a coordinate transform, we first solve the Lyapunov equation for each sub-system $\dot{\mathbf{x}} = A_m\mathbf{x}$, that is, we solve

$$A_m^T P_m + P_m A_m = -I_n \quad \text{for all } m \in \mathcal{P},$$

where I_n denotes the $n \times n$ identity matrix. Since each individual sub-system $\dot{\mathbf{x}} = A_m\mathbf{x}$ has an exponentially stable equilibrium at the origin, the matrices A_m are Hurwitz and these equations all have a positive definite solution $P_m \succ 0$. Thus, for each sub-system $\dot{\mathbf{x}} = A_m\mathbf{x}$, the function $V_m(\mathbf{x}) = \mathbf{x}^T P_m \mathbf{x}$ is a quadratic Lyapunov function. We then define the matrix

$$R := \sum_{m \in \mathcal{P}} \lambda_m P_m$$

as a convex combination of the P_m, that is, $\lambda_m \geq 0$ for all $m \in \mathcal{P}$ and $\sum_{m\in\mathcal{P}} \lambda_m = 1$. Note that R is also symmetric and positive definite, that is, $R \succ 0$.

Next, we use the coordinate transform $\mathbf{x} \mapsto R^{\frac{1}{2}}\mathbf{x}$; that is, we replace the matrices A_m by $\widetilde{A}_m := R^{\frac{1}{2}} A_m R^{-\frac{1}{2}}$. Recall that for $R \succ 0$ there is exactly one symmetric and positive definite matrix $R^{\frac{1}{2}} \succ 0$ such that $R = R^{\frac{1}{2}} R^{\frac{1}{2}}$. We denote the inverse of $R^{\frac{1}{2}}$ by $R^{-\frac{1}{2}}$ and since $R^{-\frac{1}{2}}$ is symmetric and for $\mathbf{x} \neq \mathbf{0}$ we have

$$\mathbf{x}^T R^{-\frac{1}{2}} \mathbf{x} = (R^{\frac{1}{2}}\mathbf{y})^T R^{-\frac{1}{2}} R^{\frac{1}{2}} \mathbf{y} = \mathbf{y}^T R^{\frac{1}{2}} \mathbf{y} > 0$$

where $\mathbf{y} = R^{-\frac{1}{2}}\mathbf{x} \neq \mathbf{0}$, we also have $R^{-\frac{1}{2}} \succ 0$. We refer to this procedure as *preconditioning* switched system (2). Note that if $|\mathcal{P}| = 1$ then $V(\mathbf{x}) = \mathbf{x}^T I_n \mathbf{x} =$

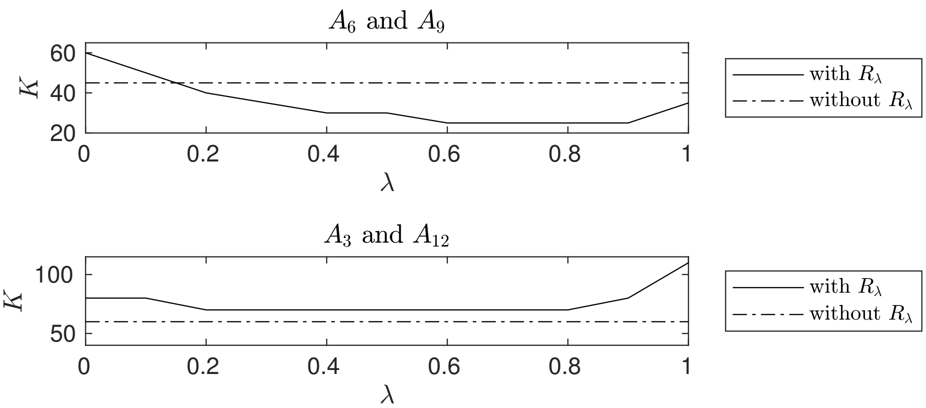


Fig. 3: Lowest resolution factor K that delivers a piecewise linear CLF as a function of λ for two examples of three-dimensional systems. The number of simplices needed is $64K^2$. While $64 \cdot 45^2 = 129,600$ simplices are needed without preconditioning, we get a solution with $64 \cdot 25^2 = 40,000$ simplices for $\lambda \in [0.6, 0.9]$ for $\mathcal{P} = \{6, 9\}$ (upper figure). For $\mathcal{P} = \{3, 12\}$, we have an atypical case, for which preconditioning is counterproductive (lower figure). The first example possesses a QCLF while the second does not. The figure is adapted from [3], where also A_m is defined for $m = 3, 6, 9, 12$.

$\|\mathbf{x}\|^2$ is a Lyapunov function for the transformed system. Furthermore, a Lyapunov function for the transformed systems is easily transformed back using the inverse $\mathbf{x} \mapsto R^{-\frac{1}{2}}\mathbf{x}$ of the coordinate transform $\mathbf{x} \mapsto R^{\frac{1}{2}}\mathbf{x}$. In Figure 2, we show how trajectories of two systems are altered through preconditioning.

In the majority of cases, one can compute a piecewise linear CLF for switched system (2) with fewer simplices when using this preconditioning. That is, one can compute a CLF using a lower value for K in the triangulation $\mathcal{T}_K$ (or $\mathcal{T}_K^{\mathbf{F}}$). However, it is unfortunately not transparent how to choose the optimal value of parameter λ_m. In Figure 3, we investigate this for two cases from the set of three-dimensional linear systems defined in the appendix of [3]. In both cases, we consider the two systems $\dot{\mathbf{x}} = A_r\mathbf{x}$ and $\dot{\mathbf{x}} = A_s\mathbf{x}$, the coordinate transform

$$R_\lambda = \lambda P_r + (1 - \lambda)P_s,$$

which corresponds to $\lambda_r = \lambda$ and $\lambda_s = 1 - \lambda$, and report the lowest resolution parameter K in the triangulation $\mathcal{T}_K$, for which we could compute a piecewise linear CLF for the system. For $r = 6$ and $s = 9$, we see the typical case where preconditioning allows us to compute a CLF using considerably fewer simplices. However, there are also cases, where preconditioning is not advantageous or even counterproductive, as it is for $r = 3$ and $s = 12$. While usually preconditioning has the effect of reducing the number of simplices needed, choosing an optimal value for parameter λ_m is an open problem. Note that this problem is of much practical value for systems of higher dimensions, where the number of simplices is the limiting factor to for computing a CLF. To better understand the effect of preconditioning, we developed the AngleAnalysis app discussed in Section 5.

3.1 Preconditioning with Rotation

As previously discussed, we first triangulate the hyper-cube $[-K, K]^n$ in the triangulation $\mathcal{T}_K$. For numerical reasons, it is advantageous to map the vertices of $\mathcal{T}_K$ using the mapping

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{F}(\mathbf{0}) = \mathbf{0}, \quad \text{and} \quad \mathbf{F}(\mathbf{x}) = \frac{\|\mathbf{x}\|_\infty}{\|\mathbf{x}\|}\mathbf{x} \quad \text{if } \mathbf{x} \neq \mathbf{0}.$$

That is, we map every vertex on the surface of the n-dimensional hyper-cube $[-K, K]^n$ to the surface of the n-dimensional hyper-sphere with radius K.[4] We refer to this new triangulation as $\mathcal{T}_K^{\mathbf{F}}$ to stress the use of mapping $\mathbf{F}$. In this paper, we use this triangulation for all computations. Thus, we often do not make the distinction and rather refer to it as $\mathcal{T}_K$. The process of mapping the vertices is visualised in Figure 4 for $n = 2$ and in Figure 5 for $n = 3$. In Figure 4 and Figure 5, we observe that the density of vertices varies in different regions. As a result, the orientation of the hyper-sphere might be critical with respect to ease of computing a CLF. Intuitively, in two dimensions, if the semi-major axis of the ellipse given by a level set of the CLF lies on the line connecting two opposite corners of the square used for triangulation, shown in the right panel of Figure 4, then the K necessary for successfully computing the CLF should be minimal. As a proof of concept we test the effect of rotating sub-systems A_m such that the resulting CLF is as just mentioned. This is visualised in Figure 6.

In the following, we explain how we compute the rotation. If matrix $P \succ 0$ defines the QCLF for an arbitrarily switched system (and obtained, for example, using the SDP approach), then we consider the eigenvalue decomposition of matrix P given by $P = U\Lambda U^T$, where Λ is diagonal and consists of the eigenvalues of P in ascending order and U is orthogonal. We then map the first column of U to the line $x_1 + x_2 + \ldots + x_n = 0$. Specifically, we apply the QR-decomposition to the vector of ones of length n, set $R = U^T Q$, and replace A_m by $\widetilde{A}_m = RA_mR^T$. Note that since both U and Q are orthogonal matrices, so are R and $R^{-1} = R^T$. Using this coordinate transform results in the area of largest curvature of the ellipsoid given by $\mathbf{x}^T P\mathbf{x} = c$ for a constant $c > 0$, or in other words the cone around the eigenvector corresponding to the smallest eigenvalue of P, to be transferred in the direction $(1, 1, \ldots, 1)$, where the simplices of the triangulation are of maximal density.

We tested this concept on the sets of two-dimensional and three-dimensional systems, for which we computed QCLF in [3]. The results are promising. The highest resolution factor in the two-dimensional case was $K = 5$ so there isn't much room for improvement. However, for only 2% of the cases the rotation was detrimental, for 88% the rotation was inconsequential, and for 10% the rotation was beneficial. In Figure 6, we can see what effect this rotation has on

[4] Note that there is a typo in [4] where the fraction is reversed.

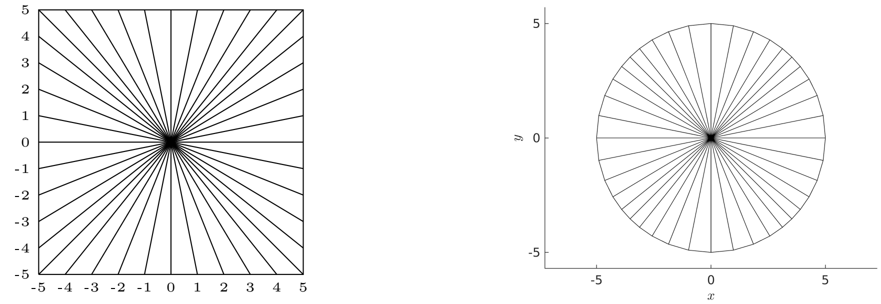


Fig. 4: The figure shows the process of creating a $\mathcal{T}_5$ triangulation in two dimensions. The triangulation of $[-5, 5]^2$ is shown in the left panel and the mapping of the square to a circle is shown in the right panel. The figure is adapted from [2].

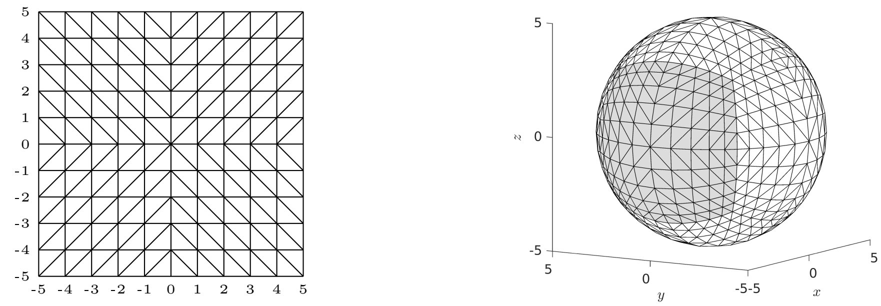


Fig. 5: The figure shows the process of creating a $\mathcal{T}_5$ triangulation in three dimensions. One face of the triangulation of $[-5, 5]^3$ is shown in the left panel and the mapping of the cube to a sphere is shown in the right panel, where the projection of the face has been coloured grey. The figure is adapted from [2].

the computed CLF when considering an arbitrarily switched system with the two-dimensional sub-systems $\dot{\mathbf{x}} = A_{14}\mathbf{x}$ and $\dot{\mathbf{x}} = A_{17}\mathbf{x}$ defined in Section 4.

The total number of three dimensional switched systems in [3] with a QCLF was fairly low, but the results still clearly show the benefit of using the rotation. In 18.2% of the cases it was detrimental, in 40.9% it was inconsequential, and in 40.9% the rotation was beneficial and reduced the number of simplices needed to compute a piecewise linear CLF. An overview of the results can be found in Table 2. There is one outlier marked by $\Delta K = -21$ for one combination of three-dimensional systems, which means that we needed a resolution parameter K for the rotated system that was 21 larger than when not using the rotation. It can be explained by the fact that two eigenvalues are much smaller than the third one and we cannot project both of them to a diagonal, where the simplices are denser. In such a case, one can get unlucky when rotating, because, although one area of large curvature is mapped to where simplices are densely distributed, other

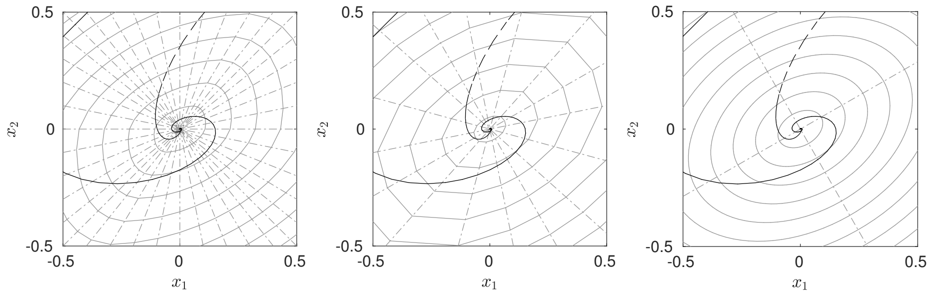


Fig. 6: Trajectories of A_{14} (black and solid), A_{17} (black and dashed), level sets of a CLF (gray and solid) and the triangulation (grey and dashed). Without rotating the coordinate system a resolution factor $K = 5$ is needed (left). When rotating before computing the CLF a resolution factor $K = 2$ is needed (center). For comparison the quadratic CLF found with SDP (right) along with the semi-minor and major axes (gray and dashed).

ones are actually mapped to areas where simplices are not densely distributed. In the majority of cases, however, the number of simplices is the same or less.

4 Computing Lyapunov Functions using SDP or LP

In this section, we compare different solvers for the established SDP approach to compute QCLF. Furthermore, we compare our LP approach from [4] to compute piecewise linear CLFs to the SDP approach. For the comparison, we construct twenty 2×2 matrices, which we denote by

$$A_{k+4(j-1)} = V_j E_k V_j^{-1}, \tag{12}$$

where $j = 1, 2, 3, 4, 5,\ k = 1, 2, 3, 4,$

$$\theta = \begin{bmatrix} 0 & \frac{9\pi}{40} & \frac{9\pi}{20} & \frac{27\pi}{40} & \frac{9\pi}{10} \end{bmatrix}, \quad d = \begin{bmatrix} 1 & 2.1544 & 4.6416 & 10 \end{bmatrix},$$

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

$$V_j = \begin{bmatrix} R(\theta_j) \begin{bmatrix} 1 \\ 1 \end{bmatrix} & R\left(\theta_j + \frac{\pi}{3}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix} \quad \text{and} \quad E_k = \begin{bmatrix} -1 & -d_k \\ d_k & -1 \end{bmatrix}.$$

We then consider the switched systems defined by (2) with these A_m and $\emptyset \neq \mathcal{P} \subseteq \mathcal{Q} = \{1, 2, \ldots, 20\}$. Note that the origin is exponentially stable for each individual system, since the eigenvalues of (12) are given by $\lambda = -1 \pm d_k \sqrt{-1}$. Moreover, they all have the same convergence rate. However, the spiralling rate is different and depends on k. System trajectories, starting at $(1, 1)$, are shown in Figure 7.

Table 2: The effect of preconditioning with rotation on the lowest resolution parameter K needed to compute a CLF for sets of two-dimensional ($n = 2$) and three-dimensional ($n = 3$) systems from [3]. $\Delta K := K_{\text{no R}} - K_{\text{with R}}$ and thus $\Delta K > 0$ means that we needed fewer simplices when we used the rotation and $\Delta K < 0$ means that we needed more simplices when we used the rotation.

ΔK	# cases $n = 2$	# cases $n = 3$
-21	$\star$	1
-4	$\star$	1
-2	$\star$	1
-1	28	1
0	1126	9
1	80	4
2	2	3
3	3	$\star$
4	40	$\star$
5	$\star$	1
14	$\star$	1
Total:	1279	22

4.1 Numerical Study: SDP

In this section, we compare different solvers and different tolerance parameters to compute QCLF for the switched system by solving LMI. We start by investigating the stability of all switched linear systems defined by (2) and all the subsets $\mathcal{P} \neq \emptyset$ of an a priori fixed superset $\mathcal{Q}$. In more detail, for each switched system corresponding to a given subset $\mathcal{P} \subseteq \mathcal{Q}$, we use SDP to search for a QCLF that solves the LMIs given by

$$P - \varepsilon I_n \succeq 0, \tag{13}$$

$$A_m^T P + P A_m + \varepsilon I_n \preceq 0, \quad \forall m \in \mathcal{P} \subseteq \mathcal{Q}. \tag{14}$$

Here, $\varepsilon > 0$ is a small constant to force positive definiteness.

To speed up the computations, we use two simple tricks. First, if there is not a solution for a given subset $\mathcal{P} \subset \mathcal{Q}$, then there cannot be a solution for any superset P^*, $\mathcal{Q} \supseteq \mathcal{P}^* \supseteq \mathcal{P}$. Second, a solution to the LMI given by (13)-(14) cannot exist for a particular subset $\mathcal{P}$ if there is a matrix in the cone defined by the matrices A_m, $m \in \mathcal{P}$, that is not Hurwitz, or more specifically, if for some choice of λ_m matrix $\widetilde{A} := \sum_{m \in \mathcal{P}} \lambda_m A_m$ is not Hurwitz. The reason for this is that

$$\widetilde{A}^T P + P \widetilde{A} \prec 0 \tag{15}$$

has a solution $P \succ 0$ if and only if $\widetilde{A}$ is Hurwitz and

$$\widetilde{A}^T P + P \widetilde{A} = \sum_{m \in \mathcal{P}} \lambda_m \left(A_m^T P + P A_m \right). \tag{16}$$

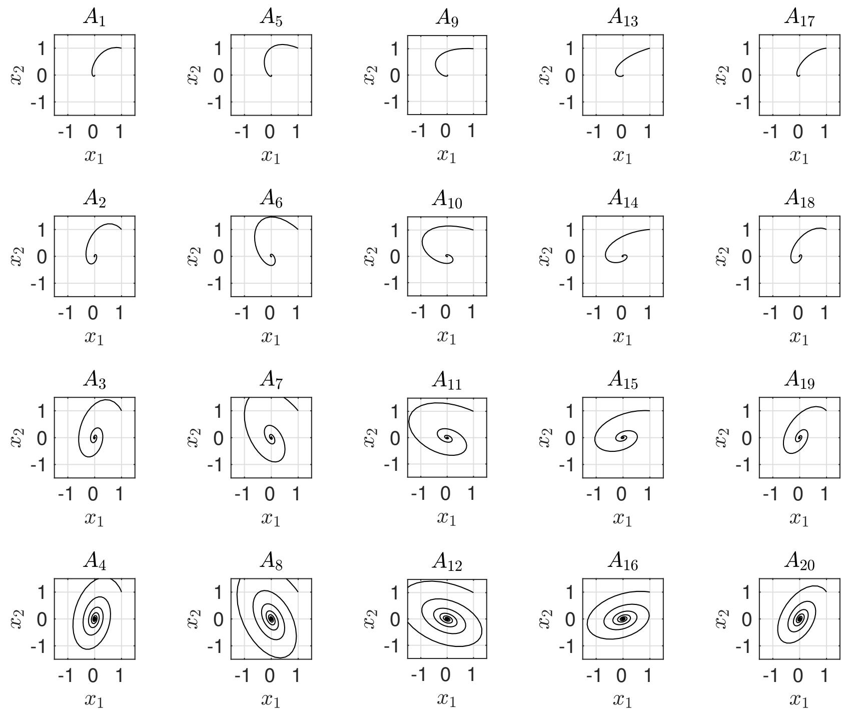


Fig. 7: Trajectories of the systems $\dot{\mathbf{x}} = A_m \mathbf{x}$, $m = 1, 2, \ldots, 20$, starting at $(1, 1)$. The matrices A_m are defined in (12). The figure is adapted from [3].

Thus, if $\widetilde{A}$ is not Hurwitz then we know that there cannot exist a solution to (13)-(14), because otherwise the left-hand-side of (16) is negative definite with the $P \succ 0$ from the solution, which is contradictory to (15) not having a solution $P \succ 0$. Hence, as a computationally very cheap test to eliminate unnecessary computations, for a subset $\mathcal{P}$, we first verify whether $\sum_{m \in \mathcal{P}} A_m$ is Hurwitz or not. If it is not Hurwitz then we do not need to attempt to solve (13)-(14) for this particular $\mathcal{P}$.

For our computations, we use the MATLAB [29] toolbox YALMIP [26] to search for a solution for all possible $\mathcal{P} \subseteq \{1, 2, \ldots, 20\}$ using the solvers SeDuMi [41], MOSEK [32], and SDPT3 [42]. For all solvers, we use their default parameters and tolerances. To identify false positives, we additionally verify that a solution reported by a solver truly satisfies (6). This is necessary because solvers sometimes report solutions to problems that do not really fulfill the constraints, particularly, if the problems are numerically badly conditioned.

For some subsets $\mathcal{P} \subseteq \mathcal{Q}$, the origin is an exponentially stable equilibrium of the corresponding switched system, for instance, if we choose only singlets. However, there are subsets $\mathcal{P}$, for which the origin is not exponentially stable. Furthermore, in some cases there are subsets $\mathcal{P}$, for which the origin is exponen-

Table 3: Overview of the results in two dimensions. PT: Solution reported and passed test, FP: False positive (solution reported but did not pass test), I: Infeasible, and T: Total number of LMIs problems. The tolerance parameter $\varepsilon = 10^{-3}$ appears reasonable and $\varepsilon = 10^{-16}$ does not. Note the effect of the simplification of not considering supersets of a set $\mathcal{P}$, for which the LMI problem is infeasible. With $\varepsilon = 10^{-3}$ only 1,366 LMIs have to be solved instead of $2^{20} - 1 = 1,047,296$, i.e. for every subsets $\mathcal{P} \neq \emptyset$ of $\mathcal{Q}$.

Solver	ε	PT	FP	I	T
MOSEK		1279	0	87	1366
SDPT3	10^{-3}	1279	0	87	1366
SeDuMi		1279	0	87	1366
MOSEK		1279	1,047,296	0	1,048,575
SDPT3	10^{-16}	1279	1,047,296	0	1,048,575
SeDuMi		1279	0	87	1366

tially stable, but there does not exist a QCLF for the corresponding switched linear system. We summarise the results of our computations in Table 3.

If we choose a sensible tolerance parameter, such as $\varepsilon = 10^{-3}$, then we obtain the same results independently of the solver used. If we choose a very low tolerance parameter such as $\varepsilon = 10^{-16}$, which is numerically close to zero, then we obtain, possibly as one would expect, more solutions. For two-dimensional systems, the results are not really useful, since the solvers MOSEK and SDPT3 report all LMIs as feasible, while solver SeDuMi delivers exactly the same results as when we use $\varepsilon = 10^{-3}$. Surprisingly however, $P \neq 0$, as one might expect for such a low tolerance. Importantly, for very badly conditioned five-dimensional systems, we showed in [3] that one obtains many more true solutions by using the very low tolerance parameter $\varepsilon = 10^{-16}$ and then verifying the solutions, than by using a more reasonable tolerance parameter $\varepsilon = 10^{-3}$. Thus, we propose to first use a tolerance that seems unreasonably low, like $\varepsilon = 10^{-16}$, and then verify the solutions obtained, since checking for false positives is computationally very cheap and we manage to obtain solutions that are missed with a more sensible tolerance parameter ε.

4.2 Numerical Study: LP vs. SDP

As we have discussed earlier in this paper, even if the origin of the switched system given by (2) is exponentially stable, a QCLF does not necessarily exist. However, the origin is an exponentially stable equilibrium for the switched system (2) if and only if there exists a piecewise linear CLF [17, 27, 28]. Moreover, a piecewise linear CLF can always be computed with our LP method from [4] as shown in [21]. Thus, we expect that by using the LP approach, we will be able to prove exponential stability of (2) for more subsets $\mathcal{P} \subseteq \mathcal{Q} = \{1, 2, \ldots, 20\}$ than by solving an LMI.

For our investigation, we compare the results from the last section, where we searched for a QCLF by means of SDP for two-dimensional systems, with the LP

Table 4: Ratio, in percentage, of the two-dimensional switched systems (1) successfully solved using SDP to problems successfully solved using our LP method, for matrices (12). Note that # refers to number of systems.

$\mathcal{Q}$	# solved (SDP)	# solved (LP)	SDP/LP	LP/tot.	total #
1	20	20	100%	100%	20
2	104	142	73.24%	74.74%	190
3	260	522	49.81%	45.79%	1,140
4	370	1092	33.88%	22.54%	4,845
5	316	1458	21.67%	9.404%	15,504
6	160	1261	12.69%	3.253%	38,760
7	44	696	6.322%	0.8978%	77,520
8	5	233	2.146%	0.1850%	125,970
9	0	42	0%	0.02501%	167,960
10	0	3	0%	0.001624%	184,756
11	0	0	$\star$	0%	167,960

Table 5: Ratio, in percentage, of the three-dimensional switched linear systems from [3] successfully solved using SDP to problems successfully solved using our LP method, where # refers to number of systems.

$\mathcal{Q}$	# solved (SDP)	# solved (LP)	SDP/LP	LP/tot.	total #
1	12	12	100%	100%	12
2	9	10	90%	15.15%	66
3	1	3	33.33%	1.36%	220
4	0	0	$\star$	0	495

approach from [4], where we search for a piecewise linear CLF using the Gurobi solver [20]. As shown in Table 4, the LP approach increasingly outperforms the SDP approach for larger $|\mathcal{P}|$. For instance, for $|\mathcal{P}| = 2$, the SDP approach guarantees exponential stability for only 75% of the systems, for which the LP approach guarantees it. For $|\mathcal{P}| = 3$, the percentage decreases to 50%, for $|\mathcal{P}| = 4$, to 33%, etc. For completeness, we also include Table 5 from [3], which shows similar results for three-dimensional systems. The results are comparable to the two-dimensional case.

5 The AngleAnalysis App

As we discussed thoroughly in Section 3, it is by no means easy to determine a good matrix R for preconditioning the switched linear system (1) for the computation of a piecewise linear CLF using LP. To offer some visual guidance for the preconditioning and in order to understand the challenge better, we developed the MATLAB application AngleAnalysis. It visualises the different local angles between two linear vector fields over the state space. That is, for the

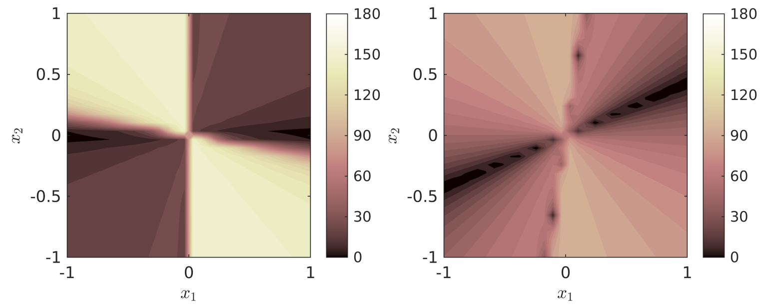


Fig. 8: Angle visualisation for matrices from [4] with $a = 0.1$ and $b = 13.25$ without preconditioning (left) and with $\lambda_a = 0.2$ and $\lambda_b = 0.8$ (right). A piecewise linear CLF was obtained for the preconditioned system using a much fewer simplices than without (lower K). The figure is adapted from [3].

systems $\dot{\mathbf{x}} = A\mathbf{x}$ and $\dot{\mathbf{x}} = B\mathbf{x}$, we visualise the angle

$$\angle(A\mathbf{x}, B\mathbf{x}) := \arccos\left(\frac{\langle A\mathbf{x}, B\mathbf{x}\rangle}{\|A\mathbf{x}\|\,\|B\mathbf{x}\|}\right) \tag{17}$$

as a function of $\mathbf{x}$, where $A, B \in \mathbb{R}^{n\times n}$, $n = 2, 3$ (see Figure 2 (a)). Note that $\langle \mathbf{x}, \mathbf{y}\rangle$ denotes the Euclidian scalar product of the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. In the following, we explain how the angle in (17) relates to the stability of the origin.

First, note that function $V : \mathbb{R}^n \to [0, \infty)$ is a Lyapunov function for the switched linear system (2) if and only if $V(\mathbf{0}) = 0$, $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$, and the directional derivative along any system trajectory is strictly negative, that is, $\langle \nabla V(\mathbf{x}), A_m\mathbf{x}\rangle < 0$ for all $m \in \mathcal{P}$ and $\mathbf{x} \neq \mathbf{0}$. It follows that for asymptotic stability of the switched system $\dot{\mathbf{x}} = A_m\mathbf{x}$, $\mathcal{P} = \{1, 2\}$, $A_1 = A$ and $A_2 := B$, the conditions $\langle \nabla V(\mathbf{x}), A\mathbf{x}\rangle < 0$ and $\langle \nabla V(\mathbf{x}), B\mathbf{x}\rangle < 0$ for all $\mathbf{x} \neq \mathbf{0}$ must hold. This implies that $\angle(\nabla V(\mathbf{x}), A\mathbf{x}) > 90°$ and $\angle(\nabla V(\mathbf{x}), B\mathbf{x}) > 90°$ must hold for all $\mathbf{x} \neq \mathbf{0}$. Then, intuitively, this condition is more difficult to satisfy if (17) tends to be large or, conversely, the smaller (17) the more "space" there is for the gradient $\nabla V(\mathbf{x})$. Hence, it should be easier to find a function $V : \mathbb{R}^n \to [0, \infty)$, whose gradient fulfils $\angle(\nabla V(\mathbf{x}), A\mathbf{x}) > 90°$ and $\angle(\nabla V(\mathbf{x}), B\mathbf{x}) > 90°$ for all $\mathbf{x} \neq \mathbf{0}$ if (17) is small for all $\mathbf{x} \neq \mathbf{0}$. However, note that this is only a local condition and even though the angle (17) is small, but not zero, for all $\mathbf{x} \neq \mathbf{0}$, this does not necessarily mean that the equilibrium is asymptotically stable for the switched system.

Given matrix $A \in \mathbb{R}^{n\times n}$ and matrix $B \in \mathbb{R}^{n\times n}$, the AngleAnalysis app visualises the angle given by (17) as a function of $\mathbf{x}$ through a heat-map, where small angles are represented by dark colours and large angles by bright colours. For $n = 2$, the heat-map is plotted in the square $[-1, 1]^2$. For $n = 3$, the heat-map is plotted in the square $[-1, 1]^2 \times \{x_3\}$, where the value of parameter x_3 is chosen by the user from the interval $[-1, 1]$.

To provide further insight, next, we look at two extreme cases. For instance, if there exists a $\hat{\mathbf{x}} \in \mathbb{R}^n$, for which (17) is 180°, then there exists a constant $c > 0$

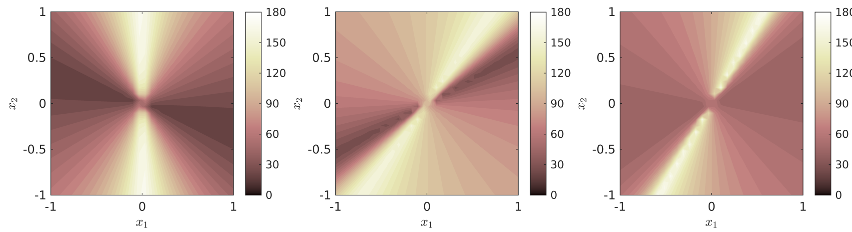


Fig. 9: Angle visualisation for A_3 and A_{12} at $x_3 = 0$. Without preconditioning (left), with $\lambda_6 = 0.7$ and $\lambda_9 = 0.3$ (center) and with $\lambda_6 = 0$ and $\lambda_9 = 1$ (right). Corresponding angle data can be found in Table 6. The figure is adapted from [3].

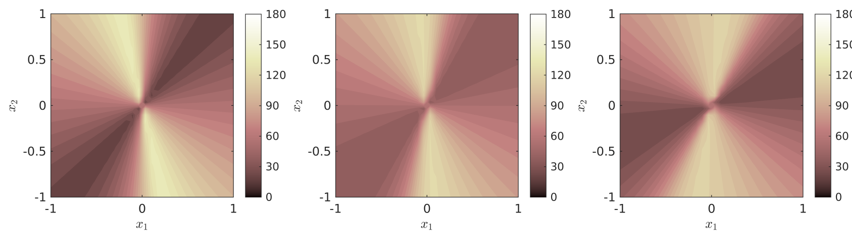


Fig. 10: Angle visualisation for A_3 and A_{12} at $x_3 = 0$. Without preconditioning (left), with $\lambda_3 = 0.7$ and $\lambda_{12} = 0.3$ (center), and with $\lambda_3 = 1$ and $\lambda_{12} = 0$ (right). Corresponding angle data can be found in Table 7. This figure is adapted from [3].

such that $A\hat{\mathbf{x}} = -cB\hat{\mathbf{x}}$ and a Lyapunov function V, such that

$$\langle \nabla V(\mathbf{x}), A\mathbf{x} \rangle < 0 \quad \text{and} \quad \langle \nabla V(\mathbf{x}), B\mathbf{x} \rangle < 0 \tag{18}$$

holds for all $\mathbf{x} \neq \mathbf{0}$, cannot exists. To see this, assume for the sake of the argument that (18) holds. Then this leads to the following contradiction:

$$0 > \langle \nabla V(\mathbf{x}), A\mathbf{x} \rangle = \langle \nabla V(\mathbf{x}), -cB\mathbf{x} \rangle = -c \langle \nabla V(\mathbf{x}), B\mathbf{x} \rangle > 0$$

for $\mathbf{x} = \hat{\mathbf{x}}$. Hence, if there exists a $\hat{\mathbf{x}} \in \mathbb{R}^n$, for which (17) is 180°, then a CLF for systems $\dot{\mathbf{x}} = A\mathbf{x}$ and $\dot{\mathbf{x}} = B\mathbf{x}$ does not exist. Another extreme case is given by (17) being 0° for all $\mathbf{x} \in \mathbb{R}^n$. Then, a Lyapunov function guarantees exponential stability of $\dot{\mathbf{x}} = A\mathbf{x}$ if and only if it guarantees exponential stability of $\dot{\mathbf{x}} = B\mathbf{x}$. In fact, $A = cB$ for some $c > 0$.

For some systems, the app clearly indicates that we can expect a reduction in the number of simplices after preconditioning, see Figure 8 and the reference [4]. In other cases the benefits of preconditioning do not become clear. For the 3×3 matrices A_m from the appendix of [3] we show in Figure 9 and Figure 10 the values of (17) by means of a heat-map with preconditioning and without preconditioning. Table 6 and Table 7 show the corresponding angle statistics.

Table 6: Maximum, minimum, mean, and standard deviation of (17) for matrix A_6 and matrix A_9 at $(x_1, x_2, x_3) \in [-1,1]^3$ where x_1, x_2, and x_3 are 30 evenly spaced points.

λ_6	λ_9	Max	Min	Mean	Std	K
N/A	N/A	176.0	2.97	77.5	43.0	45
0.7	0.3	176.5	1.02	105.3	30.4	25
0	1	176.8	0.83	82.5	37.7	60

Table 7: Maximum, minimum, mean, and standard deviation of (17) for matrix A_3 and matrix A_{12} at $(x_1, x_2, x_3) \in [-1,1]^3$ where x_1, x_2, and x_3 are 30 evenly spaced points.

λ_3	λ_{12}	Max	Min	Mean	Std	K
N/A	N/A	143.0	1.04	74.2	25.2	60
0.7	0.3	136.3	1.14	70.0	20.0	70
1	0	151.9	1.38	64.2	23.0	110

In both case, the app does not seem to provide support for deciding about the parameters in the preconditioning. The reason is that, although it is not difficult to precondition the system such that locally the gradients $\nabla V(\mathbf{x})$ fulfil $\angle(\nabla V(\mathbf{x}), A\mathbf{x}) > 90°$ and $\angle(\nabla V(\mathbf{x}), B\mathbf{x}) > 90°$, $V : \mathbb{R}^n \to [0, \infty)$ is a continuous function on $\mathbb{R}^n$ and the problem of the existence of a CLF is highly non-local.

6 Conclusions

We discussed methods to compute Lyapunov functions for switched linear systems using semidefinite programming (SDP) and linear programming (LP). We compared the SDP solvers SeDuMi, MOSEK, and SDPT3 for computing quadratic common Lyapunov functions (QCLFs) for a variety of sets of linear systems. Such common Lyapunov functions (CLFs) are also Lyapunov functions for the corresponding switched linear systems. Furthermore, we tried different tolerance parameters $\varepsilon > 0$ to force positive definiteness of matrix A through $A \succeq \varepsilon I_n$ and obtained the surprising result that using very small ε enhanced the performance of the approach in some examples. Hence, it is advisable to use a very low $\varepsilon > 0$, such as $\varepsilon = 10^{-16}$, and then verify solutions afterwards.

We compared computing QCLF with SDP to computing piecewise linear CLF using the LP approach from [4]. As expected, since switched linear systems with an asymptotically stable equilibrium at the origin do not necessarily possess a QCLF, the LP approach is able to assert stability for more switched systems than the SDP approach. In fact, it was proved in a recent publication [21] that our LP approach always succeeds in computing a CLF, if enough simplices are used in the triangulation, that is, if resolution parameter K is large enough. A drawback of the LP approach is that a triangulation of the state space is needed, which can result in a very high number of simplices in higher dimensions. We described different preconditioning methods for our LP method, including a new method based on rotations that has not been studied before, to better deal with this problem. Unfortunately, it is not clear how to choose the optimal parameters for the preconditioning methods. Finally, we developed the MATLAB app AngleAnalysis to get visual information about the effect of the preconditioning and gave examples of its use. In some cases, it gives a clear indication of which parameters are good, but in other cases this is not as clear.

In summary, we presented state-of-the art approaches to determine the asymptotic stability of the origin for switched linear systems. The traditional SDP approach is faster, but it is often not able to assert the asymptotic stability of the origin. Our LP approach is computationally more demanding, but is able to assert asymptotic stability of the origin for many more switched linear systems. In theory, it can always assert the asymptotic stability of an asymptotically stable equilibrium given sufficient computational power. For future research, it seems promising to extend the SDP approach to higher-order polynomial Lyapunov functions, as this approach is less conservative. Furthermore, as shown in this paper, preconditioning can greatly accelerate the LP approach. We will further investigate how to choose optimal parameters for the preconditioning.

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