Investigating Average Dwell-Time Constraints on Multiple Lyapunov Functions for Switched Linear Systems

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Abstract: We present a new method to compute the minimum average dwell-time needed to assert the global exponential stability of the equilibrium at the origin for a switched linear system. The method attempts to compute compatible Lyapunov functions for the individual subsystems of the switched system using linear programming or linear matrix inequalities optimization problems. We test the method on four examples to demonstrate its applicability.

Keywords: Switched systems, minimum average dwell-time, Lyapunov function, numerical method.

1. INTRODUCTION

We consider the n-dimensional switched linear system

$$\dot{\mathbf{x}} = A_{\sigma(t)}\mathbf{x}, \quad \sigma : [0,\infty) \to \mathcal{P} := \{1,2,\dots,P\}, \qquad (1)$$

where $A_i \in \mathbb{R}^{n\times n}$ and the origin is globally exponentially stable (GES) for all the individual subsystems $\dot{\mathbf{x}} = A_i\mathbf{x}$, $i \in \mathcal{P}$, and where the switching signal σ is right-continuous and only has a finite number of discontinuity-points on every finite time-interval. We are interested in the stability of the equilibrium at the origin for the switched system. Note that even though the origin is GES for all the individual subsystems, it might not be stable for the arbitrary switched system (1), see e.g. Remark 2.3 in Liberzon (2003) for a counterexample. However, by limiting the allowed rate of switching sufficiently, the origin will always be GES for the switched system (1).

An appropriate concept to discuss the allowed rate of switching is the so-called average dwell-time, see Hespanha and Morse (1999) or, e.g., Chapter 3.2 in Liberzon (2003). We say that a switching signal σ has an average dwell-time $\tau > 0$, if there exists $N_0 > 0$, such that

$$N_\sigma(t,s) \le N_0 + \frac{t-s}{\tau} \quad \text{for all } t \ge s \ge 0,$$

where $N_\sigma(t,s)$ denotes the number of discontinuity-points of σ on the open interval (s,t), i.e. $N_\sigma(t,s)$ is the number of times that σ switches value between times s and t. The set of all switching signals with average dwell-time τ is denoted by Σ_τ and we talk about the switched system (1) with average dwell-time τ, if σ in (1) can be an arbitrary element from Σ_τ. When we say that the origin is GES for (1) with average dwell-time τ, we mean that there exist constants $C \ge 1$ and $\beta > 0$, such that

$$\|\mathbf{x}(t)\| \le C\,\mathrm{e}^{-\beta t}\,\|\mathbf{x}(0)\|$$

for all solutions $\mathbf{x}(\cdot)$ to (1) with $\sigma \in \Sigma_\tau$; $\|\cdot\|$ denotes the Euclidian norm.

⋆ This work was supported in part by the Icelandic Research Fund under Grant 228725-051.

An interesting question is now:

> What is the minimum average dwell-time τ, such that the origin is GES for the switched system (1)?

A sufficient condition is given in terms of Lyapunov functions for the subsystems in the following theorem adapted from Theorem 3.2 in Liberzon (2003); recall that $\mathcal{K}_\infty$ is the set of strictly increasing, unbounded, continuous functions $[0,\infty) \to [0,\infty)$, that are zero at zero.

Theorem 1. Assume that for each $i \in \mathcal{P}$, there exists a locally Lipschitz continuous function $V_i\colon \mathbb{R}^n \to \mathbb{R}$, such that

$$\underline{a}^*(\|\mathbf{x}\|) \le V_i(\mathbf{x}) \le \overline{a}^*(\|\mathbf{x}\|), \quad \forall i \in \mathcal{P}, \qquad (2a)$$

$$D^+V_i(\mathbf{x}, A_i\mathbf{x}) \le -\alpha V_i(\mathbf{x}), \quad \forall i \in \mathcal{P}, \qquad (2b)$$

$$V_i(\mathbf{x}) \le \mu V_j(\mathbf{x}), \quad \forall i,j \in \mathcal{P}, \qquad (2c)$$

where $\overline{a}^*, \underline{a}^* \in \mathcal{K}_\infty$, $\alpha > 0$, and $\mu \ge 1$. Then the origin is GES for the switched system (1) with average dwell-time τ, if τ satisfies

$$\tau > \frac{\ln(\mu)}{\alpha}. \qquad (3)$$

Here,

$$\begin{aligned} D^+V_i(\mathbf{x}, A_i\mathbf{x}) &:= D^+(V_i\circ\mathbf{x})(t)\big|_{t=0} \\ &:= \limsup_{h\to 0+} \frac{V_i(\mathbf{x}(t+h)) - V_i(\mathbf{x}(t))}{h}\bigg|_{t=0} \\ &= \limsup_{h\to 0+} \frac{V_i(\mathbf{x}+hA_i\mathbf{x}) - V_i(\mathbf{x})}{h} \end{aligned}$$

is the Dini-derivative of V_i along the solution trajectories of the system $\dot{\mathbf{x}} = A_i\mathbf{x}$. The proof can be copied almost verbatim from the proof of Theorem 3.2 in Liberzon (2003), although our V_i are only locally Lipschitz continuous and we use the Dini-derivative instead of the usual derivative, because with $W(t) := \mathrm{e}^{\alpha t}\, V_{\sigma(t)}(\mathbf{x}(t))$ we have for two consecutive discontinuity times t_j and t_{j+1} of σ for every $t_j \le t < t_{j+1}$, that

$$
\begin{aligned}
W(t) &= W(t_j) + \int_{t_j}^{t} D^+W(\tau)\,\mathrm{d}\tau \\
&= W(t_j) + \int_{t_j}^{t} \Big[\alpha\, \mathrm{e}^{\alpha\tau}\, V_{\sigma(t_j)}(\mathbf{x}(\tau)) \\
&\qquad + \mathrm{e}^{\alpha\tau}\, D^+V_{\sigma(t_j)}(\mathbf{x}(\tau), A_{\sigma(t_j)}\mathbf{x}(\tau))\Big]\,\mathrm{d}\tau \\
&\le W(t_j) \quad \text{by (2b)},
\end{aligned}
$$

see e.g. Theorem 1 in Hagood and Thomson (2006).

The following corollary to Theorem 1 was used in Hafstein and Tanwani (2023) to develop a method to compute a lower bound on the minimum average dwell-time τ for the switched system (1), such that the origin is GES.

Corollary 2. Assume that for each $i \in \mathcal{P}$, there exists a locally Lipschitz continuous functions $V_i\colon \mathbb{R}^n \to \mathbb{R}$, such that

$$
\begin{aligned}
\underline{a}\,\|\mathbf{x}\|^d \le V_i(\mathbf{x}) &\le \overline{a}\,\|\mathbf{x}\|^d, \quad \forall i \in \mathcal{P}, \\
D^+V_i(\mathbf{x}, A_i\mathbf{x}) &\le -\alpha\|\mathbf{x}\|^d, \quad \forall i \in \mathcal{P}, \\
V_i(\mathbf{x}) &\le \mu V_j(\mathbf{x}), \quad \forall i, j \in \mathcal{P},
\end{aligned} \tag{4}
$$

$\overline{a}, \underline{a}, \alpha, d > 0$, and $\mu \ge 1$. Then the origin is GES for the switched system (1) with average dwell-time τ for every

$$
\tau > \frac{\overline{a}\ln(\mu)}{\alpha}. \tag{5}
$$

Note that in the corollary the $\mathcal{K}_\infty$ functions $\underline{a}^*$ and $\overline{a}^*$, have been replaced with the $\mathcal{K}_\infty$ functions $\mathbf{x} \mapsto \underline{a}\|\mathbf{x}\|$ and $\mathbf{x} \mapsto \overline{a}\|\mathbf{x}\|$ for constants $0 < \underline{a} \le \overline{a}$. The corollary follows immediately from $-\|\mathbf{x}\|^d \le -V_i(\mathbf{x})/\overline{a}$.

It is shown in Hafstein and Tanwani (2023), that with $d = 1$ and $d = 2$, one can fix $\underline{a}, \overline{a} > 0$, and $\mu \ge 1$, and then use optimization to maximize $\alpha > 0$, such that the conditions of Corollary 2 are fulfilled. From formula (5) one then obtains a lower bound on the minimum average dwell-time needed to assert GES of the origin for the switched system (1).

For $d = 2$ this was done by searching for quadratic Lyapunov functions $V_i(\mathbf{x}) = \mathbf{x}^T P_i \mathbf{x}$, $i \in \mathcal{P}$, by solving the linear matrix inequality (LMI) opimazation problem:

$$
\begin{aligned}
&\text{maximize } \alpha \\
&\text{subject to} \\
&\underline{a}I \preceq P_i \;\text{ and }\; P_i \preceq \overline{a}I, \quad \forall i \in \mathcal{P}, && \text{(6a)} \\
&A_i^T P_i + P_i A_i \preceq -\alpha I, \quad \forall i \in \mathcal{P}, \\
&P_i \preceq \mu P_j, \quad \forall i, j \in \mathcal{P}. && \text{(6b)}
\end{aligned}
$$

In these formulas I denotes the $n \times n$ identity matrix and $A \preceq B$ means that the matrix $B - A \in \mathbb{R}^{n\times n}$ is symmetric and positive semidefinite. The variables of the LMI optimization problem are $\alpha > 0$ and the symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{P}$.

For $d = 1$, and this was the main contribution of Hafstein and Tanwani (2023), linear programming (LP) was used to parameterize piecewise linear Lyapunov functions V_i fulfilling the conditions of Corollary 2. In the three examples presented, the LP approach with $d = 1$ outperformed the LMI approach with $d = 2$, which is not surprising because the set of piecewise linear Lyapunov functions is richer than the set of quadratic Lyapunov functions.

The advantage of using the conditions of Corollary 2 on the Lyapunov functions V_i, rather than the conditions in Theorem 1, is that the former is linear in the optimization variables while the latter is bilinear. Hence, α can be maximized using efficient and well-understood methods; LP in the case of $d = 1$ and semidefinite programming (SDP) in the case $d = 2$.

However, there is some conservatism involved in the conditions (4) in Corollary 2 with respect to the conditions (2b) in Theorem 1. Further, even after the maximum α has been found using a fixed $\mu \ge 1$, the corresponding lower bound τ in formula (3), on the needed average dwell-time to assert GES, might be suboptimal. Hence, in this paper we will derive and study methods to find low values for the average dwell-time τ using the parameters α and μ directly. We will use LP and LMI optimization problems, strongly based on those in Hafstein and Tanwani (2023), to construct Lyapunov functions, but in addition we will investigate methods to improve the values of α and μ in order to obtain lower τ.

The paper is organized as follows. In Section 2 we discuss how to parameterize compatible Lyapunov functions for the individual subsystems of the switched system (1) using LP or LMI optimization problems. In Section 3 we describe our method to compute a minimum average dwell-time τ that asserts the origin is GES for the system (1) and in Section 4 we apply our method to four examples and discuss the results, before we conclude the paper in Section 5.

2. PARAMETERIZING LYAPUNOV FUNCTIONS

The LMI optimization problem we use to construct quadratic Lyapunov functions is a straightforward adaptation of the LMI optimization problem from Hafstein and Tanwani (2023) discussed in the last section. For fixed $0 < \underline{a} < \overline{a}$, $\alpha > 0$, and $\mu \ge 1$, find symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{P}$, fulfilling

<u>LMI feasibility problem:</u>

$$
\begin{aligned}
&\underline{a}I \preceq P_i \;\text{ and }\; P_i \preceq \overline{a}I, \quad \forall i \in \mathcal{P}, && \text{(7a)} \\
&A_i^T P_i + P_i A_i + \alpha P_i \preceq 0, \quad \forall i \in \mathcal{P}, && \text{(7b)} \\
&P_i \preceq \mu P_j, \quad \forall i, j \in \mathcal{P}. && \text{(7c)}
\end{aligned}
$$

If a solution exists to the LMI optimization problem (7), then the functions $V_i(\mathbf{x}) = \mathbf{x}^T P_i \mathbf{x}$, $i \in \mathcal{P}$, are Lyapunov functions to the subsystems $\dot{\mathbf{x}} = A_i\mathbf{x}$ fulfilling $V_i(\mathbf{x}) \le \mu V_j(\mathbf{x})$ for all $i, j \in \mathcal{P}$. Hence, by Theorem 1, $\tau = \ln(\mu)/\alpha$ is a lower bound on the needed average dwell-time to assert GES of the origin for the switched system (1).

The LP problem to construct piecewise linear Lyapunov functions V_i for the switched system (1) is more involved, mainly because one needs to triangulate the state-space $\mathbb{R}^n$. As we are essentially using the construction from Hafstein and Tanwani (2023), we only give a short description and refer the reader to Hafstein and Tanwani (2023) for the details. We use the triangulation $\mathcal{T}_K^{\mathbf{F}}$, where $K \in \mathbb{N}$ is a parameter determining the density or fineness of the triangulation, which consists of the n-simplices $\mathfrak{S}_\nu = \operatorname{co}\{\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \dots, \mathbf{x}_n^\nu\}$. Here ν is just an index to enumerate the simplices $\mathfrak{S}_\nu \in \mathcal{T}_K^{\mathbf{F}}$ and $\operatorname{co}\mathcal{A}$ denotes the convex hull of the vectors in $\mathcal{A} \subset \mathbb{R}^n$. Two different simplices $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}_K^{\mathbf{F}}$ intersect in a common face, which might be as small as $\{\mathbf{0}\}$, and the set-theoretic union of the simplices is a neighbourhood of the origin. As discussed in Andersen et al. (2024) the triangulation $\mathcal{T}_K^{\mathbf{F}}$ has $2^n K^{n-1} n!$ simplices.

To every simplex $\mathfrak{S}_\nu = \text{co}\{\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu\}$ we associate the variables $V^i_{\mathbf{x}_j^\nu} \in \mathbb{R}$, $i \in \mathcal{P}$ and $j = 1, 2, \ldots, n$; note that if $\mathbf{y} = \mathbf{x}_j^\nu = \mathbf{x}_k^\mu$ for two different simplices $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$, then $V^i_{\mathbf{y}} = V^i_{\mathbf{x}_j^\nu} = V^i_{\mathbf{x}_k^\mu}$. Once the values of the variables have been fixed, we can define the functions V_i on

$$\begin{aligned}\mathfrak{C}_\nu &:= \text{cone}\{\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu\} \\ &= \left\{ \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu \ : \ \lambda_j \geq 0 \ \text{ for } \ j = 1, 2, \ldots, n \right\}\end{aligned}$$

through $V_i(\mathbf{x}_j^\nu) := V^i_{\mathbf{x}_j^\nu}$ for $j = 1, 2, \ldots, n$, and

$$V_i\left(\sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu\right) := \sum_{j=1}^n \lambda_j V^i_{\mathbf{x}_j^\nu} = \sum_{j=1}^n \lambda_j V_i(\mathbf{x}_j^\nu). \tag{8}$$

Since $\mathfrak{S}_\nu$ is an n-simplex the vertices $\mathbf{x}_j^\nu$, $j = 1, 2, \ldots, n$, are linearly independent and V_i is a well-defined linear function on $\mathfrak{C}_\nu$. Further, since two different simplices $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ intersect in a common face, V_i is well-defined and continuous on the set-theoretic union of the $\mathfrak{C}_\nu$, which is the whole of $\mathbb{R}^n$, because the union of the $\mathfrak{S}_\nu$ is a neighbourhood of the origin. Another useful equivalent formula for V_i on $\mathfrak{C}_\nu$ is

$$V_i(\mathbf{x}) = [\mathbf{v}_\nu^i]^T X_\nu^{-1} \mathbf{x},$$

where

$$[\mathbf{v}_\nu^i]^T := \begin{bmatrix} V^i_{\mathbf{x}_1^\nu} & V^i_{\mathbf{x}_2^\nu} & \cdots & V^i_{\mathbf{x}_n^\nu} \end{bmatrix} \in \mathbb{R}^{1\times n} \tag{9}$$

and

$$X_\nu := [\mathbf{x}_1^\nu \ \mathbf{x}_2^\nu \ \cdots \ \mathbf{x}_n^\nu] \in \mathbb{R}^{n\times n},$$

i.e. the $\mathbf{x}_j^\nu$ are the columns of X_ν; see e.g. Andersen et al. (2023) or Remark 9 in Giesl and Hafstein (2014) for this formula.

Given the triangulation $\mathcal{T}_K^F$, the LP feasibility problem to construct Lyapunov functions V_i for the switched system (1) is: for every simplex $\mathfrak{S}_\nu = \text{co}\{\mathbf{0}, \mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu\}$ we have the constraints for every $i, k \in \mathcal{P}$ and every $j = 1, 2, \ldots, n$:

<u>LP feasibility problem:</u>

$$\underline{a}\,\|\mathbf{x}_j^\nu\| \leq V^i_{\mathbf{x}_j^\nu} \leq \overline{a}\,\|\mathbf{x}_j^\nu\|, \tag{10a}$$

$$[\mathbf{v}_\nu^i]^T X_\nu^{-1} A_i \mathbf{x}_j^\nu \leq -\alpha [\mathbf{v}_\nu^i]^T X_\nu^{-1} \mathbf{x}_j^\nu, \tag{10b}$$

$$V^i_{\mathbf{x}_j^\nu} \leq \mu V^k_{\mathbf{x}_j^\nu}, \tag{10c}$$

Theorem 3. Any solution to the LP feasibility problem (10) delivers Lyapunov functions $V_i \colon \mathbb{R}^n \to \mathbb{R}$, $i \in \mathcal{P}$, through the formulas (8), that fulfill (2) with appropriate $\underline{a}^*, \overline{a}^* \in \mathcal{K}_\infty$.

Proof: Follows immediately from formulas (8) and (9) and the fact that for every $\mathbf{x} \in \mathbb{R}^n$ and $i \in \mathcal{P}$, there exists a $\mathfrak{C}_\nu$, such that $\mathbf{x} + hA_i\mathbf{x} \in \mathfrak{C}_\nu$ for all small enough $0 \leq h$; hence by (10b) and with $\mathbf{x} = \sum_{j=1}^n \lambda_j \mathbf{x}_j^\nu$, $\lambda_j \geq 0$, we have

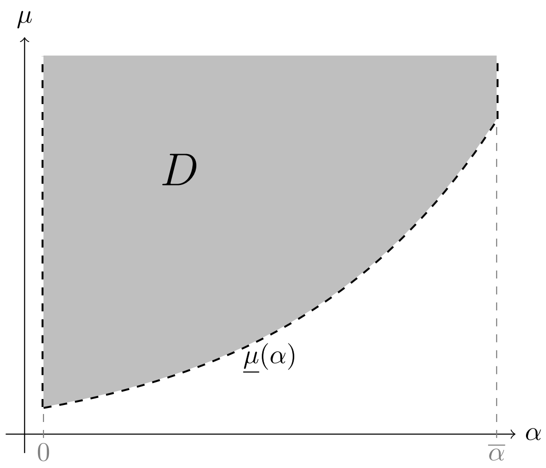


Fig. 1. An example of the area D where (2) has a feasible solution given α and μ. Note that the area extends indefinitely as $\mu \to \infty$. The function $\alpha \mapsto \underline{\mu}(\alpha)$ is discussed at the beginning of Section 4. All of our computations indicate that $\underline{\mu}(\alpha)$ is convex, although we have not been able to prove this statement.

$$\begin{aligned} D^+V_i(\mathbf{x}, A_i\mathbf{x}) &= [\mathbf{v}_\nu^i]^T X_\nu^{-1} A_i \mathbf{x} \\ &= \sum_{j=1}^n \lambda_j [\mathbf{v}_\nu^i]^T X_\nu^{-1} A_i \mathbf{x}_j^\nu \\ &\leq -\alpha \sum_{j=1}^n \lambda_j [\mathbf{v}_\nu^i]^T X_\nu^{-1} \mathbf{x}_j^\nu \\ &= -\alpha\, [\mathbf{v}_\nu^i]^T X_\nu^{-1} \mathbf{x} \\ &= -\alpha\, V_i(\mathbf{x}). \end{aligned}$$

□

3. MINIMIZING THE MINIMUM AVERAGE DWELL-TIME

For the switched system (1) define

$$D := \{(\alpha, \mu) \in (0, \infty) \times [1, \infty) \colon (2) \text{ is feasible}\}, \tag{11}$$

see Figure 1 for a schematic picture. By Theorem 1 the switched system (1) is GES with average dwell-time $\tau > \ln(\mu)/\alpha$ for every $(\alpha, \mu) \in D$. To obtain the optimal minimum average dwell-time that asserts GES for the system (1), we attempt to

$$\text{minimize} \quad \tau := \tau(\alpha, \mu) := \frac{\ln(\mu)}{\alpha} \quad \text{on } D.$$

We will approximate the set D using two classes for the Lyapunov functions V_i. In more detail, we use the LP problem (10) to compute piecewise linear Lyapunov functions and the LMI optimization problem (7) to compute quadratic Lyapunov functions. Hence, checking whether a point (α, μ) is in D or not involves solving a feasibility problem. Now one can of course proceed with a brute force approach and check for numerous points on a dense grid if $(\alpha, \mu) \in D$, but since this is quite expensive numerically, we will suggest a more clever way to search systematically for (α, μ) that deliver low values for τ. Note that since τ is nonlinear and a function of two variables, it is not trivial figuring out which parameter or parameters to focus on in its minimization. Methods like gradient descent are a poor option for optimizing both α and μ at the same time, because the update

$$\begin{bmatrix}\alpha_{k+1}\\ \mu_{k+1}\end{bmatrix} = \begin{bmatrix}\alpha_k\\ \mu_k\end{bmatrix} - \eta\nabla\tau(\alpha_k,\mu_k),$$

with the learning rate $\eta > 0$, does not consider the set D and τ obtains its minimum $\tau = 0$ for every $(\alpha, 1)$, $\alpha \in \mathbb{R}$.

Before we suggest our method to minimize τ in Algorithm 1, we first make the following useful observation:

3.1 Upper bound on α

Recall that a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ is GES, if and only if A is Hurwitz. We can use this to our advantage and compute upper bounds on potential α, such that the LMI optimization problem (7) or the LP problem (10) are feasible.

LMI: The Lyapunov inequality $A_i^T P_i + P_i A_i \prec 0$ has a symmetric positive definite solution $P_i \succ 0$, if and only if A_i is Hurwitz, see Sylvester (1884). The inequality (7b) implies

$$\begin{aligned} 0 &\succeq A_i^T P_i + P_i A_j + \alpha P_i \\ &= A_i^T P_i + P_i A_i + \frac{\alpha}{2}P_i + \frac{\alpha}{2}P_i \\ &= \left(A_i + \frac{\alpha}{2}I\right)^T P_i + P_i\left(A_i + \frac{\alpha}{2}I\right) \end{aligned}$$

That is, for (7b) to be feasible, the real parts of the eigenvalues of $A_i + \frac{\alpha}{2}I$ must be strictly negative for all $i \in \mathcal{P}$. This gives us an upper bound $\overline{\alpha}_{\mathrm{LMI}}$ on α for the LMI optimization problem (7).

LP: Similarly, consider that (10b) implies

$$\begin{aligned} 0 &\geq [\mathbf{v}_\nu^i]^T X_\nu^{-1} A_i \mathbf{x}_j^\nu + \alpha[\mathbf{v}_\nu^i]^T X_\nu^{-1}\mathbf{x}_j^\nu \\ &= [\mathbf{v}_\nu^i]^T X_\nu^{-1}(A_i + \alpha I)\mathbf{x}_j^\nu. \end{aligned}$$

That is, for (10b) to be feasible, the real parts of the eigenvalues of $A_i + \alpha I$ must be strictly negative for all $i \in \mathcal{P}$. This gives us an upper bound $\overline{\alpha}_{\mathrm{LP}}$ on α for the LP problem (10).

3.2 Our method to minimize the average dwell-time

We suggest the following algorithm to search for points $(\alpha,\mu) \in D$ that minimize τ. Recall, that we either use the LP problem (10) to search for piecewise linear Lyapunov functions V_i or the LMI optimization problem (7) to search for quadratic Lyapunov functions V_i. Thus, we first fix the method to search for feasibility as either the LP problem (10) or the LMI optimization problem (7). The method is referred to as the *feasibility problem* with values (α,μ). If we are using the LP problem (10) we define $\overline{\alpha} := \overline{\alpha}_{\mathrm{LP}}$ and if we are using the LMI optimization problem (7) we define $\overline{\alpha} := \overline{\alpha}_{\mathrm{LMI}}$. Before the algorithm is executed, it is worthwhile to first check the feasibility for $(\alpha,\mu) = (0,1)$. If it is feasible then one usually also gets a solution for a small $\alpha > 0$ and $\mu = 1$, which shows GES for the switched system (1) for arbitrary switchings. That is, no minimum average-dwell time is needed for stability.

Algorithm 1. We fix $N, M \in \mathbb{N}$, $N \geq 2$, and $a, b > 0$ and distribute the values $a = \tau_N < \tau_{N-1} < \ldots < \tau_2 < \tau_1 = b$ and $0 < \alpha_1 < \ldots < \alpha_M < \overline{\alpha}$ uniformly, i.e.

$$\tau_j = (a-b)\frac{j-1}{N-1} + b \quad \text{and} \quad \alpha_i = \overline{\alpha}\frac{i}{M+1}$$

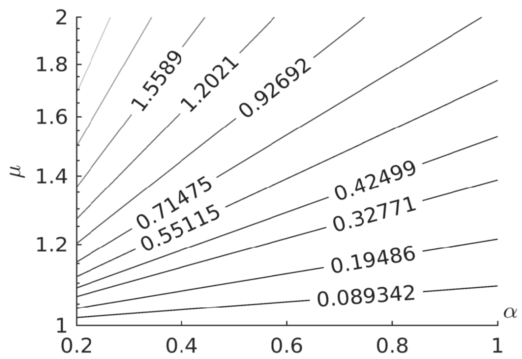


Fig. 2. Level sets for $\tau = \ln(\mu)/\alpha$ on $[0.2, 1] \times [1, 2]$. Note the μ-axis is logarithmical, therefore each level set is linear in the figure.

for $j = 1, 2, \ldots, N$ and $i = 1, 2, \ldots, M$.

Set $i = 1$ and $j = 1$. Then execute:

(1) Solve the feasibility problem with $\alpha = \alpha_i$ and $\mu = \exp(\tau_j\alpha_i)$.

(2)
- If there is a feasible solution to the problem, then increase j by one, i.e. $j \leftarrow j+1$, and go back to step (1). That is, we decrease τ from τ_j to τ_{j+1} and check if this problem, with a lower average dwell-time and a smaller $\mu = \exp(\tau_{j+1}\alpha_i)$, also has a feasible solution.
- If there does not exists a feasible solution to the problem, then increase i by one, i.e. $i \leftarrow i+1$, and go back to step (1). That is, we increase α from α_i to α_{i+1} and set $\mu = \exp(\tau_j\alpha_{i+1})$ and check if this problem, with the same average dwell-time $\tau = \tau_j$, but with larger parameters α and μ, has a feasible solution.

Note that the algorithm follows the level sets of τ, see Figure 2 for an exemplary picture, as long as the optimization problem is not feasible. If the optimization problem is feasible we jump to the next level-set below for the same α and try our luck.

4. EXAMPLES

To visually interpret the examples the following nomenclature is useful: We say that the set D in (11) is *approximated by the function* $\alpha \mapsto \underline{\mu}(\alpha)$ if graph$(\underline{\mu}) \subset D$. Note that if $(\alpha,\mu^*) \in D$, then $(\alpha,\mu) \in D$ for all $\mu \geq \mu^*$. Hence, the epigraph $\{(\alpha,\mu) \in (0,\infty)\times[1,\infty) \colon \mu \geq \underline{\mu}(\alpha)\}$ is a subset of D. In the following we explicitly write the argument α in $\underline{\mu}(\alpha)$ to emphasize that $\underline{\mu}$ is a function of α.

For each example we computed $\underline{\mu}(\alpha)$ using both the LP problem (10) and the LMI optimization problem (7) with a brute forced approach; checking the feasibility of the optimization problem on a dense grid in the (α,μ) plane. Starting with $(\alpha,\mu) = (0,1)$, where the optimization problem is not feasible, we show how Algorithm 1 first approaches $\underline{\mu}(\alpha)$ and then walks along it until a minimum is found. The LP problem was implemented with C++ and solved with Gurobi (Gurobi Optimization, LLC (2024)). The LMI optimization problem was implemented in Matlab (The MathWorks Inc. (2023)) using YALMIP Löfberg (2004) and solved with SDPT3 Toh et al. (1999).

Example 1 The switched system (1) with

$$A_1 = \begin{bmatrix} -0.1 & -1 \\ 2 & -0.1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -0.1 & -2 \\ 1 & -0.1 \end{bmatrix}, \tag{12}$$

taken from (Liberzon, 2003, p. 26). Progression of Algorithm 1 using LMI and LP optimization problems are shown in Figures 3a and 4a, respectively. Using the LMI optimization we obtain the minimum average dwell-time $\tau_{\mathrm{LMI}} = 3.5195$ and using the LP optimization we obtain $\tau_{\mathrm{LP}} = 3.5636$, compared to $\tau_{\mathrm{LMI}} = 4.5283$ and $\tau_{\mathrm{LP}} = 4.6241$ in Hafstein and Tanwani (2023).

Example 2 The switched system (1) with

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -1 & -10 \\ 0.1 & -1 \end{bmatrix} \tag{13}$$

taken from Dayawansa and Martin (1999). Progression of Algorithm 1 using LMI and LP optimization problems are shown in Figures 3b and 4b, respectively. Using the LMI optimization we obtain the minimum average dwell-time $\tau_{\mathrm{LMI}} = 1.1532$ and using the LP optimization we obtain $\tau_{\mathrm{LP}} = 0$, compared to $\tau_{\mathrm{LMI}} = 17.0394$ and $\tau_{\mathrm{LP}} = 0$ in Hafstein and Tanwani (2023).

Example 3 The switched system (1) with

$$A_1 = \begin{bmatrix} -5 & 1 & 2 \\ 0 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 3 \\ -2 & -1 & -3 \\ -1 & 0 & -2 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} -4 & 0 & -3 \\ 2 & -2 & 4 \\ 1 & 0 & -1 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ -3 & 0 & -4 \end{bmatrix} \tag{14}$$

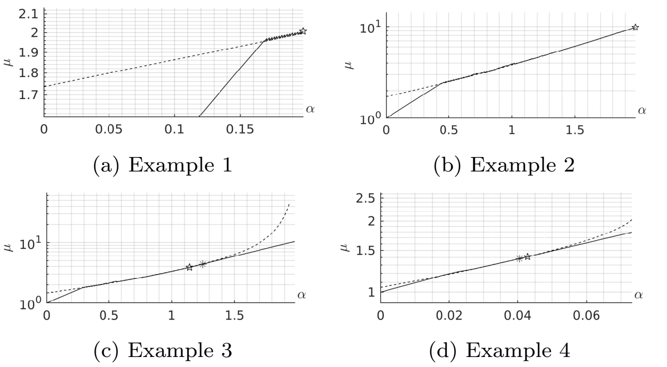
taken from Liu et al. (2021). Progression of Algorithm 1 using LMI and LP optimization problems are shown in Figures 3c and 4c, respectively. An interesting observation: $\underline{\mu}(\alpha)$ is identical to the worst case when only considering two subsystems at a time. Using the LMI optimization we obtain the minimum average dwell-time $\tau_{\mathrm{LMI}} = 1.1852$ and using the LP optimization we obtain $\tau_{\mathrm{LP}} = 0$, compared to $\tau_{\mathrm{LMI}} = 4.6870$ and $\tau_{\mathrm{LP}} = 0$ in Hafstein and Tanwani (2023), both of which are improvements over the approach proposed in Liu et al. (2021).

Example 4 The switched system (1) with

$$A_1 = \begin{bmatrix} -0.5302 & 0.0012 & 0.0873 \\ 0.2185 & -0.7494 & 0.5411 \\ 0.7370 & 0.1543 & -0.3606 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.5136 & 0.4419 & 0.3689 \\ 0.1840 & -0.3951 & 0.0080 \\ 0.3163 & 0.6099 & -1.0056 \end{bmatrix}. \tag{15}$$

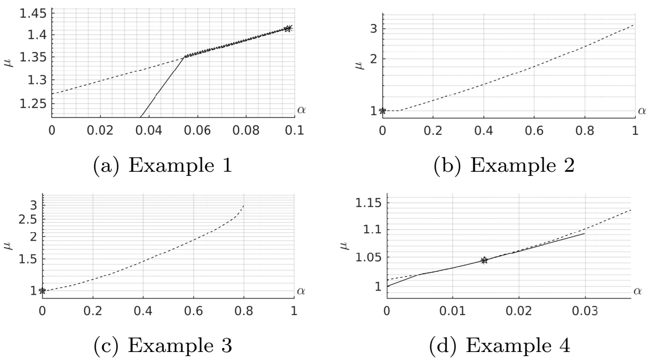
Progression of Algorithm 1 using LMI and LP optimization problems are shown in Figures 3d and 4d, respectively. Using the LMI optimization we obtain the minimum average dwell-time $\tau_{\mathrm{LMI}} = 8.0090$ and using the LP optimization we obtain $\tau_{\mathrm{LP}} = 3.0305$,



(a) Example 1 (b) Example 2

(c) Example 3 (d) Example 4

Fig. 3. Progression of Algorithm 1 for the switched system (1) with the matrices A_i from (12) (a), (13) (b), (14) (c), and (15) (d) using the LMI optimization problem (7). The dashed lines show $\underline{\mu}(\alpha)$ computed with a brute force approach. Marked with an asterisk is the location of the smallest τ on $\underline{\mu}(\alpha)$ and marked with a pentagram is the smallest τ found with Algorithm 1.



(a) Example 1 (b) Example 2

(c) Example 3 (d) Example 4

Fig. 4. Progression of Algorithm 1 for the switched system (1) with the matrices A_i from (12) (a), (13) (b), (14) (c) and (15) (d) using the LP optimization problem (10). The dashed lines show $\underline{\mu}(\alpha)$. Marked with an asterisk is the location of the smallest τ on $\underline{\mu}(\alpha)$ and marked with a pentagram is the smallest τ found with Algorithm 1.

4.1 Discussion of the Results

Note that for both two dimensional systems (Example 1 and Example 2) we get $\underline{\mu}(\alpha) \approx e^{C_1 \alpha} + C_2$ for constants C_1 and C_2, whenever $\underline{\mu}(0) \neq 1$, posing the question; can we find a closed form solution for $\underline{\mu}(\alpha)$ in the two dimensional case? Another observation is that whenever $\underline{\mu}(0) \neq 1$, the smallest τ is obtained when $\alpha \to \overline{\alpha}$.

When examining the three dimensional systems (Example 3 and Example 4) this apparent consistency disappears; the location of the best τ is no longer predictable nor is the shape of $\underline{\mu}(\alpha)$.

Figures 3 and 4 show how Algorithm 1 follows $\underline{\mu}(\alpha)$. In Table 1 we can see that Algorithm 1 finds the same τ as the brute force method with, on average, an error of 4×10^{-2}. As expected the LP problems generally return lower minimum average-dwell times, with the exception of Example 1. Note that the minimum of τ is also a function of K, i.e. the resolution of

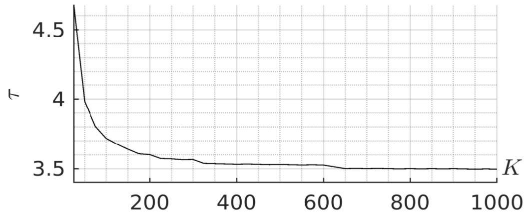


Fig. 5. The minimum average dwell-time τ obtained using the LP optimization problem (10) as a function of the resolution of the triangulation K used.

the triangulation used, and in Figure 5 we depict this dependency for Example 1. For large K the minimum value of τ approaches the minimum value obtained with the LMI optimization problem, indicating that in this particular example quadratic Lyapunov functions are optimal.

Note that the brute force computations done for comparison are computationally very demanding and can only be done practically for small optimization problems. For example, in the Brute Force computations for Example 1 we solved 4,800 optimization problems, in comparison to 200 optimization problems when using Algorithm 1.

Table 1. Results of Algorithm 1 compared to the results from using a brute force approach and the method used in Hafstein and Tanwani (2023), denoted Method (∗).

Ex.	Method	Optimization	τ	α	μ
1	Brute Force	SDP	3.4180	0.1996	1.9960
	Algorithm 1		3.5195	0.1980	2.0074
	Method (∗)		5.1929	N/A	2
	Brute Force	LP ($K = 300$)	3.5401	0.9790	1.4142
	Algorithm 1		3.5636	0.0860	1.3995
	Method (∗)	LP ($K = 500$)	4.5283	N/A	1.4
2	Brute Force	SDP	1.1516	1.9990	9.9946
	Algorithm 1		1.1532	1.9800	9.8084
	Method (∗)		17.039	N/A	3.1
	Brute Force	LP ($K = 300$)	0	*	1
	Algorithm 1		0	*	1
	Method (∗)		0	*	1
3	Brute Force	SDP	1.1845	1.2440	4.3647
	Algorithm 1		1.1852	1.1400	3.8617
	Method (∗)		4.6870		2.7
	Brute Force	LP($K = 6$)	0	*	1
	Algorithm 1		0	*	1
	Method (∗)		0	*	1
4	Brute Force	SDP	7.9994	0.0405	1.3829
	Algorithm 1		8.0090	0.0428	1.4090
	Brute Force	LP($K = 25$)	2.9451	0.0146	1.0440
	Algorithm 1		3.0305	0.0059	1.0229

5. CONCLUSION

We described a new method to compute the minimum average dwell-time τ needed to assert the global exponential stability (GES) of the equilibrium at the origin for a switched linear system. For this, the method solves linear programming (LP) or linear matrix inequality (LMI) optimization problems as in Hafstein and Tanwani (2023), but searches for the optimal parameters (α, μ) directly, instead of fixing μ and maximizing α as in Hafstein and Tanwani (2023). We demonstrated our approach on four examples and made some interesting observations. In all cases we obtain lower values for τ than reported in the literature.

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