Manuscript submitted to AIMS Journal

doi:10.3934/xx.xxxxxxx

CPA LYAPUNOV FUNCTIONS FOR TIME-PERIODIC SYSTEMS IN ONE SPATIAL DIMENSION

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(Communicated by Handling Editor)

Abstract. Continuous piecewise affine (CPA) functions have been successfully used to construct Lyapunov functions for autonomous ordinary differential equations with an exponentially stable equilibrium. However, for time-periodic systems this method cannot be used. In this paper, we develop a method to compute a Lyapunov function for a time-periodic ordinary differential equation with a (known) exponentially stable periodic solution. While the usual approach with a triangulation works away from the periodic solution, a new approach is required in a neighborhood of the periodic solution. We derive sufficient conditions for a Lyapunov function and, moreover, we prove a converse theorem, showing that these conditions can always be satisfied if the triangulation is sufficiently fine. Finally, we formulate the construction problem as a linear programming problem and apply it to two examples. This paper deals with the case of one spatial dimension, but we believe that the method can be generalised to any dimension, which will be done in a subsequent paper.

1. **Introduction.**

The determination of attractors and their basins of attraction in dynamical systems is a very important task in applications. However, for nonlinear systems, this is a challenge and can mostly not be achieved by analytical methods. Therefore, many numerical methods have been developed.

One of the most successful methods is the method of Lyapunov functions. A Lyapunov function V is a scalar-valued function, which has a minimum at the attractor and is decreasing along trajectories of the system. Lyapunov functions were introduced in [19] as a tool to determine the stability of equilibria of systems of differential equations. The method of Lyapunov functions has been extended to different systems such as nonautonomous systems [14, 23, 18], arbitrary switched autonomous systems [13] and differential inclusions [4]. The classical definition of a (strict) Lyapunov function requires the function to have a minimum on the attractor (in our case the periodic orbit) and the orbital derivative, the derivative along solutions, to be strictly negative apart from the attractor, see e.g. [18, Theorem 4.10] for time-periodic systems. There are Lyapunov functions with weaker sufficient conditions, e.g. the decrease of the function is only required after a fixed time-step, see e.g. [1] and [18, Theorem 8.5]. However, as these conditions are difficult to verify in practice and an exponentially stable periodic orbit implies the existence of

2020 *Mathematics Subject Classification.* Primary: 37M22, 38C60; Secondary: 37C75.

Key words and phrases. Time-periodic dynamical system, Lyapunov function, Continuous-piecewise affine function, Basin of attraction, Converse theorem.

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a Lyapunov function satisfying the classical definition, as we will show in a converse theorem, we will use the classical definition.

Converse theorems establish the existence of such a Lyapunov function and have been shown in a variety of contexts. For dynamical systems, given by a linear right-hand side $\dot{x} = Ax$, the asymptotic stability of the equilibrium at the origin is equivalent to the existence of a quadratic Lyapunov function, which can be found solving a linear matrix equation, the so-called Lyapunov equation. For nonlinear systems with an exponentially stable equilibrium, we can thus compute a Lyapunov function for the linearised system, which is a local Lyapunov function for the nonlinear system, i.e. a Lyapunov function in a neighborhood of the equilibrium. However, if we want a reasonable estimate on the basin of attraction, or investigate stability for more complicated systems, then we need to look for more advanced methods to find Lyapunov functions.

The review [10] gives an overview over numerical methods to construct Lyapunov functions; let us consider three of these numerical methods: the SOS method, the CPA method and the RBF method. All three methods are designed to generate a Lyapunov function for an autonomous system with exponentially stable equilibrium at the origin.

The Sum of Squares method (SOS), developed by Parrilo in [22], analyses dynamical systems $\dot{x} = f(x)$, where f is a polynomial. Then, this method introduces a special type of polynomial called 'SOS polynomial'. These polynomials are constructed as the sum of squared terms. By using computers, it is possible to find a Lyapunov function V for $\dot{x} = f(x)$ that is a SOS polynomial. By construction, V satisfies all the conditions of a Lyapunov function and hence proves the stability of the equilibrium point for the entire domain of V. For an overview of the SOS method, we refer the reader to [3, 2].

Next, the RBF (radial basis functions) method is a general collocation method that solves linear PDEs [5]. It uses radial basis functions to numerically solve the Zubov equation. Finally, there is the CPA (continuous and piecewise affine) method, which we will use in this paper and which will be explained in detail; CPA Lyapunov functions for equilibria of autonomous systems have been studied in [20, 6, 9], but the origins go back to [17, 16].

For the CPA method for an equilibrium consider the domain of a non-linear system $\dot{x} = f(x)$, $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$. First, we need to divide the domain of f into simplices; Figure 1 (left) shows a triangulation into simplices in two dimensions. A CPA Lyapunov function V is determined by the values at the vertices and is defined as the affine interpolation on each simplex.

In [9] it is proven that for any such system $\dot{x} = f(x)$ with an exponentially stable equilibrium, it is always possible to find a Lyapunov function using this CPA method. To do this, the authors present a modified triangulation method with a triangular fan around the equilibrium, see Figure 1 (right).

In this paper we will consider a time-periodic system $\dot{x} = f(t, x)$, where $f(t + T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R}^2$, for some period T. We assume that $x(t) = 0$ is a (periodic) solution of the system, i.e. $f(t, 0) = 0$ for all $t \in \mathbb{R}$. In general, if we know a periodic orbit of the dynamical system, we can then modify the system so that the periodic orbit is at zero. To obtain a CPA function with minimum on the periodic orbit, it is necessary to choose a triangulation such that the periodic orbit, in this case the t-axis, is the union of 1-simplices.

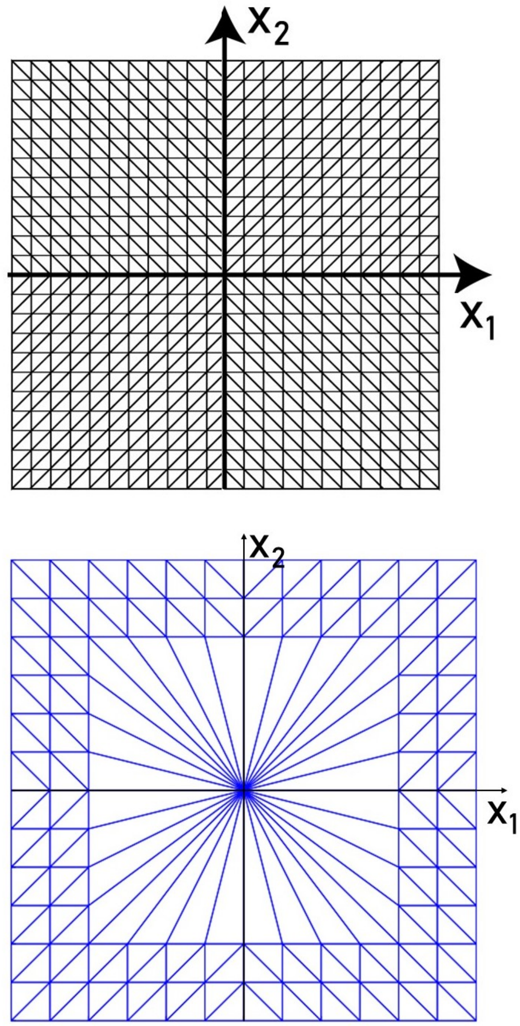

Figure 1. Left: Standard triangulation of $\mathbb{R}^2$. Right: Modified triangulation with “fan” at the origin.

The main contribution of this paper is to develop the CPA method for time-periodic systems with periodic orbits. We will establish sufficient conditions for the existence of a CPA Lyapunov function for a periodic orbit and also prove a converse theorem, ensuring that these conditions can always be fulfilled for a sufficiently fine triangulation. In more detail, Theorem 4.1 shows that a Lyapunov function exists for a time-periodic system with exponentially stable periodic solution at zero, while Theorem 4.3 proves the existence of a CPA Lyapunov function, which can be verified by checking that Constraints 3.2 and 3.12 are satisfied, if the triangulation is sufficiently fine. We will, however, need to introduce a new way to define the Lyapunov function near the periodic orbit, since the traditional CPA method and triangulation does not work for these systems.

Let us explain why the triangulation method does not work for time-periodic systems in general. Consider the triangulation in Figure 1 (left), see Figure 2 for detail in the (t, x)-plane. A CPA Lyapunov function would have to have a minimum along the zero solution. If we assume that this minimum is 0, then we require $V(t, 0) = 0$ for all t and $V(t, x) > 0$ for all $x \neq 0$.

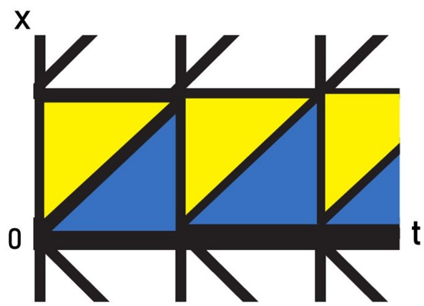

Figure 2. Zoom of the general triangulation method

Figure 2 shows the two types of triangles we get from the general triangulation of the domain: yellow triangles have only one vertex on the $x = 0$ line (where the zero solution is located), and we can choose the values at the other two vertices such that the CPA Lyapunov function decreases within the yellow triangles. The blue triangles, however, have two vertices (t_0, x_0), (t_1, x_1) with $x_0 = x_1 = 0$ on the $x = 0$ line and the value at those two vertices is necessarily $V(t_0, x_0) = V(t_1, x_0) = 0$. The third vertex (t_2, x_2) satisfies $x_2 > 0$. Hence, we have from $(t, x) = \sum_{i=0}^{2} \lambda_i (t_i, x_i) = (\sum_{i=0}^{2} \lambda_i t_i, \lambda_2 x_2)$ with $\lambda_i \geq 0$ and $\sum_{i=0}^{2} \lambda_i = 1$ that $\lambda_2 = \frac{x}{x_2}$.

For any choice $V(t_2, x_2) = c > 0$ of the value at the last vertex we have

$$V(t, x) \quad = \quad \sum_{i=0}^{2} \lambda_i V(t_i, x_i) \quad = \quad \lambda_2 c \quad = \quad \frac{xc}{x_2}.$$

Now consider the system $\dot{x} = f(t, x) = -(1 - \sin(t))x$ with initial condition $x(t_0) = \xi$, which has the solution $x(t) = \xi \exp(t_0 + \cos(t_0) - t - \cos(t))$. As the solution fulfills

$$|x(t)| = \exp(\cos(t_0) - \cos(t)) \exp(-(t - t_0))|\xi| \leq e^2 e^{-(t-t_0)} |\xi|$$

the zero solution $x(t) = 0$ is exponentially stable. However, since $f(t, x) = 0$ for $t = (1/2 + 2k)\pi$, $k \in \mathbb{Z}$, the CPA function $V(t, x)$ above cannot be a Lyapunov function for the system because on a blue triangle where $t_0 < \pi/2 \leq t_1$ we have $V(t, x) = xc/x_2$ and for $t = \pi/2$ we get

$$\dot{V}(t, x) = V_t(t, x) + V_x(t, x) f(t, x) = 0 + \frac{c}{x_2} f(\pi/2, x) = 0,$$

where $V_t := \frac{\partial V}{\partial t}$ and $V_x := \frac{\partial V}{\partial x}$.

Hence, we propose a new type of parameterization of the Lyapunov function near the periodic orbit. We divide the domain $D \subset S_T \times \mathbb{R}$ under consideration into a neighborhood $R_0 = S_T \times [-x^*, x^*]$ of the 0 solution and its complement; in the complement we will use a classical CPA function with a suitable triangulation, while we will define V differently in R_0. Figure 3 shows the division into R_0 and $D \setminus R_0$, the latter with the standard triangulation.

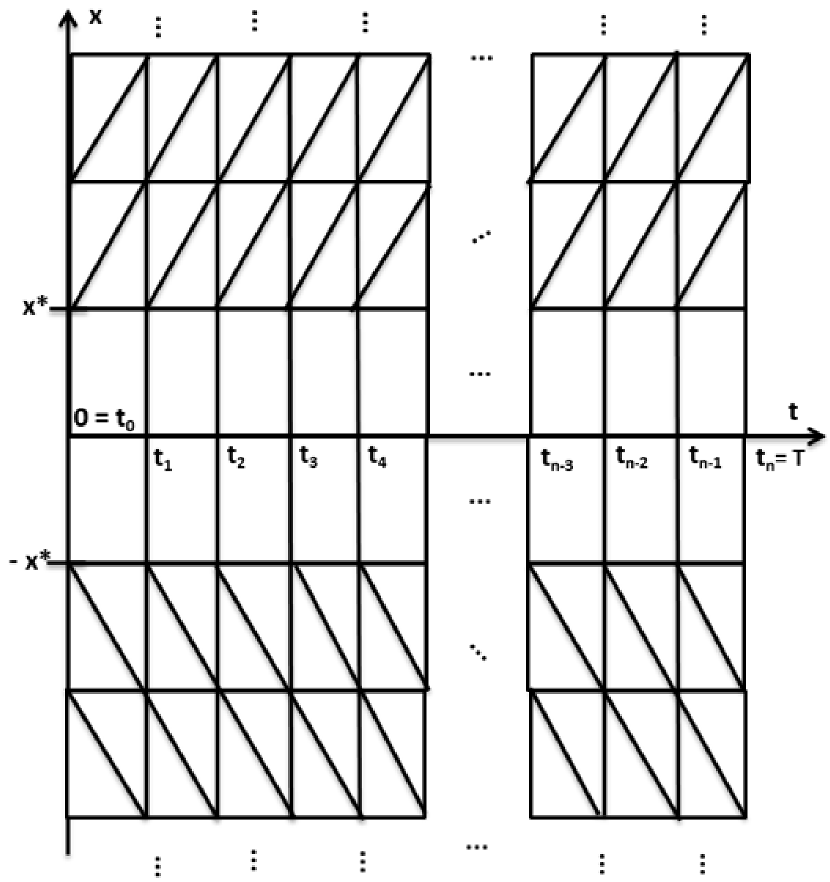

FIGURE 3. Definition of a CPA Lyapunov function for the time-periodic system $\dot{x} = f(t, x)$. In $R_0 := [0, T] \times [-x^*, x^*]$, the Lyapunov function will be defined on each rectangle, while in the complement, the Lyapunov function is a CPA function, defined on a standard triangulation with triangles as shown.

We choose $0 = t_0 < t_1 < \ldots < t_n = T$ and define a Lyapunov function in a rectangle $[t_i, t_{i+1}] \times [0, x^*]$ (case $x > 0$) or $[t_i, t_{i+1}] \times [-x^*, 0]$ (case $x < 0$) in R_0 as follows:

$$V(t, x) = \frac{|x|}{x^*} \sum_{j=0}^{1} \lambda_j V(t_{i+j}, \pm x^*), \tag{1}$$

where $t = \sum_{j=0}^{1} \lambda_j t_{i+j}$ with $\sum_{j=0}^{1} \lambda_j = 1$ and $1 \geq \lambda_j \geq 0$ and $V(t_{i+j}, \pm x^*)$ are given values of V at the vertices.

V is defined in $D \setminus R_0$ in the classical way, i.e. for a triangle S_ν with vertices (t_i^ν, x_i^ν), $i = 0, 1, 2$, we set

$$V(t, x) = \sum_{j=0}^{2} \lambda_j V(t_j^\nu, x_j^\nu), \tag{2}$$

where $(t,x) = \sum_{j=0}^{2} \lambda_j (t_j^\nu, x_j^\nu)$ with $\sum_{i=0}^{2} \lambda_j = 1$ and $1 \geq \lambda_j \geq 0$. To make sure that the function V overall is well defined and continuous, we require that the vertices at the boundary ∂R_0 are the same for the rectangles and the triangles.

Let us give an overview over the paper: in Section 2 we introduce time-periodic systems and Lyapunov functions for periodic solutions of such systems. In Section 3 we give sufficient conditions for a Lyapunov function for time-periodic systems, while in Section 4 we show a converse theorem, proving that the conditions can be fulfilled if the triangulation is sufficiently fine. Section 5 applies the method to two examples before we conclude. This paper is partly based on the PhD thesis of the third author [15].

2. **Time-Periodic Dynamical System.** In this paper we will consider time-periodic dynamical systems in one spatial dimension.

Definition 2.1. Consider the non-linear equation

$$\dot{x} = f(t,x), \tag{3}$$

where $f \in C^3(\mathbb{R}^2, \mathbb{R})$ and $f(t+T,x) = f(t,x)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}$. We assume that the solution $x(t)$ of the initial value problem $x(t_0) = \xi_0$ exists for all $t \geq t_0$ and we denote

$$\begin{aligned} \phi^x(t;t_0,\xi_0) &:= x(t) \\ \phi(t;t_0,\xi_0) &:= (t \bmod T, x(t)) \end{aligned}$$

Then $(t,(t_0,\xi)) \mapsto \phi(t;t_0,\xi)$ defines a (semi-)dynamical system on $S_T \times \mathbb{R}$, where S_T denotes the circle of circumference T. A periodic orbit is given by

$$\{\phi(t;0,\xi) \mid t \in [0,T)\}$$

for $\xi \in \mathbb{R}$ such that $\phi^x(T;0,\xi) = \xi$.

Remark 2.2. If $\phi^x(t;0,\xi) = p(t)$ is a (known) periodic orbit, then we can modify the system (2.1) such that the periodic orbit is the zero solution by using $y(t) = x(t) - p(t)$, for which the ODE

$$\dot{y} = \tilde{f}(t,y) := f(t, y + p(t)) - f(t, p(t))$$

holds.

In the following, we will thus assume that our time-periodic system has the zero solution, i.e. $f(t,0) = 0$ for all t. Its stability and basin of attraction in $S_T \times \mathbb{R}$ can be determined using a Lyapunov function.

Definition 2.3. A T-periodic strict Lyapunov function for the zero solution of $\dot{x} = f(t,x)$ as in Definition 2.1 with $f(t,0) = 0$ for all $t \in S_T$ is a locally Lipschitz function $V \colon D \to \mathbb{R}$, where $S_T \times \{0\} \subset D^\circ \subset S_T \times \mathbb{R}$ which satisfies

1. $V(t+T,x) = V(t,x)$ for all $(t,x) \in D$ (by definition),
2. $V(t,0) = 0$ for all t and $V(t,x) > 0$ for all $(t,x) \in D$ with $x \neq 0$,
3. $D_+V(t,x) < 0$ for all $(t,x) \in D^\circ$ with $x \neq 0$.

Here, $D_+V(t,x)$ denotes the Dini-derivative as defined in [6, Definition 2.3]:

$$\begin{aligned} D_+V(t,x) &= \limsup_{h \to 0+} \frac{V(\phi(t+h;t,x)) - V(\phi(t;t,x))}{h} \\ &= \limsup_{h \to 0+} \frac{V(t+h, \phi^x(t+h;t,x)) - V(t,x)}{h}. \end{aligned}$$

Theorem 2.4. *Let V be a Lyapunov function as in Definition 2.3 for the system (2.1). Then the zero solution is asymptotically stable and any compact set $V^{-1}([0,A])$ with $A>0$, which lies in D° is a subset of its basin of attraction.*

The theorem can be proved by standard methods of the Lyapunov theory, e.g. by adapting the proof of [21, Theorem 1.16] to time-periodic systems.

3. **Sufficient conditions for a Lyapunov function.** In this section, we will formulate conditions that imply the existence of a Lyapunov function. We will first consider a neighborhood R_0 of the zero solution, where the Lyapunov function will be defined on each rectangle of the form $[t_i,t_{i+1}]\times[0,x^*]$ or $[t_i,t_{i+1}]\times[-x^*,0]$. Later, we will triangulate $D\setminus R_0^\circ$ and define a CPA Lyapunov function as usual, which will be compatible to the local definition and thus result in a continuous function on D.

As mentioned in the introduction, the classical definition of CPA functions is not suitable to define a Lyapunov function near the periodic orbit. Hence, we will propose a different type of Lyapunov function locally.

Before we define the Lyapunov function, we use the fact that $f(t,0)=0$ for all $t\in S_T$ to write the right-hand side in the form $f(t,x)=xg(t,x)$.

Lemma 3.1. *Let $f\in C^3(\mathbb{R}^2,\mathbb{R})$, $f(t+T,x)=f(t,x)$ for all $(t,x)\in\mathbb{R}$ and $f(t,0)=0$ for all $t\in[0,T]$. Then*

$$g(t,x)=\int_0^1 f_x(t,\tau x)d\tau, \tag{4}$$

is in $C^2(\mathbb{R}^2,\mathbb{R})$, satisfies $g(t+T,x)=g(t,x)$ for all $(t,x)\in\mathbb{R}$ and $xg(t,x)=f(t,x)$ for all $(t,x)\in\mathbb{R}^2$.

Proof. For the path $\tau\mapsto(t,\tau x)$ from $(t,0)$ to (t,x) we have by the fundamental theorem of calculus

$$f(t,x)=f(t,x)-f(t,0)=\int_0^1 f_x(t,\tau x)\cdot x\,d\tau=g(t,x)x.$$

□

We will define the Lyapunov function on the compact set $D\subset S_T\times\mathbb{R}$, where $S_T\times\{0\}\subset D^\circ$. Let $x^*>0$ be such that $R_0=S_T\times[-x^*,x^*]\subset D^\circ$. We divide the domain R_0 into rectangles, while the area outside R_0 will be triangulated by triangles (simplices), see Figure 3.

3.1. **Lyapunov Function in R_0 near the periodic orbit.** In this part, we will consider $R_0=S_T\times[-x^*,x^*]$ and split it into rectangles. Let $0=t_0<t_1<t_2<\ldots<t_n=T$ and consider the function g from Lemma 3.1. We will assume that the values $V_{(t_i,\pm x^*)}$ at the points $(t_i,\pm x^*)$, respectively, for $i=0,\ldots,n$ satisfy the Constraints 3.2; then we will use these values to construct a Lyapunov function in R_0 in Theorem 3.4.

Constraints 3.2. Let $f\in C^3(\mathbb{R}^2,\mathbb{R})$, $f(t+T,x)=f(t,x)$ for all $(t,x)\in\mathbb{R}$, $f(t,0)=0$ for all $t\in[0,T]$ and g be defined as in Lemma 3.1; by that lemma, g_t and g_{tt} are continuous and time-periodic functions, and thus the maxima on compact sets below are well defined and finite.

Let $0=t_0<t_1<t_2<\ldots<t_n=T$, $x^*>0$ and

$$\ddot g_{\max} = \max_{t\in[0,T]}\max_{x\in[-x^*,x^*]}|g_{tt}(t,x)|$$

$$\begin{aligned} \dot{g}_{\max} &= \max_{t\in[0,T]} \max_{x\in[-x^*,x^*]} |g_t(t,x)| \\ g_{\max}(i,+) &= \max_{x\in[0,x^*]} g(t_i,x) \text{ for all } i\in\{0,\ldots,n-1\}, \\ g_{\max}(i,-) &= \max_{x\in[-x^*,0]} g(t_i,x) \text{ for all } i\in\{0,\ldots,n-1\}, \\ E_i &= (t_{i+1}-t_i)^2(C\ddot{g}_{\max}+2D\dot{g}_{\max}) \text{ for all } i\in\{0,\ldots,n-1\}. \end{aligned}$$

Given $V_{(t_i,\pm x^*)}$ for all $i \in \{0,1,2,\ldots,n\}$, and C, $D \in \mathbb{R}$, we assume

1. $V_{(t_i,\pm x^*)} \geq x^*$ for all $i \in \{0,1,2,\ldots,n\}$,
2. $V_{(t_i,\pm x^*)} \leq C$ for all $i \in \{0,1,2,\ldots,n\}$,
3. $V(t_0,\pm x^*) = V(t_n,\pm x^*)$.
4. $\left|\frac{V_{(t_{i+1},\pm x^*)}-V_{(t_i,\pm x^*)}}{t_{i+1}-t_i}\right| \leq D$ for all $i \in \{0,1,2,\ldots,n-1\}$,
5. $\frac{V_{(t_{i+1},\pm x^*)}-V_{(t_i,\pm x^*)}}{t_{i+1}-t_i} + g_{\max}(i,\pm)V_{(t_i,\pm x^*)} + E_i \leq -x^*$, and
$\frac{V_{(t_{i+1},\pm x^*)}-V_{(t_i,\pm x^*)}}{t_{i+1}-t_i} + g_{\max}(i+1,\pm)V_{(t_{i+1},\pm x^*)} + E_i \leq -x^*$
for all $i \in \{0,1,2,\ldots,n-1\}$.

Now, we can define a function $V(t,x)$ as follows:

Definition 3.3. Given R_0 with $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$, $x^* > 0$ as above and $V_{(t_i,\pm x^*)}$ for all $i \in \{0,1,2,\ldots,n\}$ with $V_{(t_0,\pm x^*)} = V_{(t_n,\pm x^*)}$, we can define a function $V\colon R_0 \to \mathbb{R}$ as follows.

Let $(t,x) \in R_0$, $|x| \leq x^*$. Then there exists $i \in \{0,1,2,\ldots,n-1\}$ such that $t_i \leq t \leq t_{i+1}$ and thus we can write $t = \sum_{j=0}^{1} \lambda_j t_{i+j}$ with $\sum_{j=0}^{1} \lambda_j = 1$ and $1 \geq \lambda_j \geq 0$. Then, we define the function $V(t,x)$ as follows:

$$V(t,x) = \frac{|x|}{x^*}\sum_{j=0}^{1} \lambda_j V_{(t_{i+j},\pm x^*)}, \tag{5}$$

where $\pm$ is determined by the sign of x; note that $V(t,0) = 0$.

The function V is well defined and continuous on R_0 and its restriction to each rectangle $[t_i,t_{i+1}] \times [0,x^*]$ or $[t_i,t_{i+1}] \times [-x^*,0]$ is C^∞.

The next theorem shows that if the values $V_{(t_i,\pm x^*)}$ satisfy the Constraints 3.2, then the function V as in Definition 3.3 is a Lyapunov function for the system $\dot{x} = f(t,x)$ on R_0.

Theorem 3.4. *Consider the system $\dot{x} = f(t,x) = xg(t,x)$, with $f \in C^3(S_T \times \mathbb{R},\mathbb{R})$ and $f(t,0) = 0$ for all t. Assume the values $V_{(t_i,\pm x^*)}$ satisfy Constraints 3.2.*

Then the function $V(t,x)$ from Definition 3.3 is a Lyapunov function for this system satisfying:

1. *$V(t,0) = 0$ for all $t \in S_T$,*
2. *$V(t,x) \geq |x|$ for all $(t,x) \in R_0$,*
3. *Denote by $V\big|_{(i,\pm)}$ the restriction of V to the rectangle $[t_i,t_{i+1}] \times [0,x^*]$ or $[t_i,t_{i+1}] \times [-x^*,0]$. For each such rectangle we have*
$$\dot{V}\big|_{(i,\pm)}(t,x) \leq -|x| \quad \text{for all } (t,x) \text{ in the rectangle.}$$

Before we prove Theorem 3.4, we compute the derivatives of the function $V(t,x)$ on each rectangle.

Lemma 3.5. *For $V(t,x)$ from Definition 3.3, restricted to the rectangle $[t_i,t_{i+1}] \times [0,x^*]$ (case $+$) or $[t_i,t_{i+1}] \times [-x^*,0]$ (case $-$), $i \in \{0,\ldots,n-1\}$, we have*

1. $V_t(t,x) = \frac{|x|}{x^*}\frac{V_{(t_{i+1},\pm x^*)}-V_{(t_i,\pm x^*)}}{t_{i+1}-t_i}$,

2. $V_x(t,x) = \frac{\pm\sum_{j=0}^{1}\lambda_j V_{(t_{i+j},\pm x^*)}}{x^*}$ *for* $x \neq 0$,

where $t = \sum_{j=0}^{1}\lambda_j t_{i+j}$ *with* $\sum_{j=0}^{1}\lambda_j = 1$ *and* $\lambda_j \geq 0$ *for* $j = 0, 1$.

Proof. We fix a rectangle $[t_i, t_{i+1}] \times [0, x^*]$ (case $+$) or $[t_i, t_{i+1}] \times [-x^*, 0]$ (case $-$), and compute the derivative at a point (t,x) in this rectangle. The values for λ_0, λ_1 as defined above are

$$\lambda_0 = \frac{t_{i+1}-t}{t_{i+1}-t_i} \quad \text{and} \quad \lambda_1 = \frac{t-t_i}{t_{i+1}-t_i}.$$

We first compute

$$\begin{aligned} V_t(t,x) &= \frac{d}{dt}\frac{|x|}{x^*}\sum_{j=0}^{1}\lambda_j V_{(t_{i+j},\pm x^*)} \\ &= \frac{|x|}{x^*}\frac{d}{dt}\left(\frac{t_{i+1}-t}{t_{i+1}-t_i}V_{(t_i,\pm x^*)} + \frac{t-t_i}{t_{i+1}-t_i}V_{(t_{i+1},\pm x^*)}\right) \\ &= \frac{|x|}{x^*}\frac{V_{(t_{i+1},\pm x^*)}-V_{(t_i,\pm x^*)}}{t_{i+1}-t_i} \end{aligned}$$

The x-derivative at $x \neq 0$ is given by

$$V_x(t,x) = \frac{d}{dx}\frac{|x|}{x^*}\sum_{j=0}^{1}\lambda_j V_{(t_{i+j},\pm x^*)} = \frac{\pm 1}{x^*}\sum_{j=0}^{1}\lambda_j V_{(t_{i+j},\pm x^*)},$$

where the sign corresponds to the rectangle. This shows the lemma. □

The next remark links the Dini derivative to the derivative of V on each of the rectangles or simplices and enables us to prove the statement about the Dini derivative by considering the derivative of V restricted to each simplex/rectangle.

Remark 3.6. If a subset $D \subset S_T \times \mathbb{R}^n$ is subdivided into a finite number of convex sets D^ν and the restrictions $V^\nu := V\big|_{D^\nu}$ of a locally Lipschitz function $V\colon D \to \mathbb{R}$ fulfill that V^ν is differentiable and

$$\dot{V}^\nu(t,x) = V_t^\nu(t,x) + V_x^\nu(t,x)\cdot f(t,x) \leq -\|x\|$$

on each D^ν, then $D^+V(t,x) \leq -\|x\|$ for every $(t,x) \in D^\circ$, because for every $(t,x) \in D^\circ$ there are D^ν and $h^* > 0$ such that $(t+h, x+hf(t,h)) \in D^\nu$ for all $0 \leq h \leq h^*$. For a proof of this fact see e.g. [7, Theorem 2.3].

We are now ready to prove Theorem 3.4.

Proof. [of Theorem 3.4] The first condition is satisfied by Definition 3.3.

To prove 2., let (t,x) be a general point on the rectangle determined by $(i,\pm)$ for $i = 0, 1, 2, \ldots, n-1$. Then we can write $(t,x) = \sum_{j=0}^{1}\lambda_j(t_{i+j}, x)$ for $\lambda_0, \lambda_1 \in [0,1]$ with $\sum_{j=0}^{1}\lambda_j = 1$. Also, note that by the first constraint we have $V_{(t_i,\pm x^*)} \geq x^*$. Then,

$$V(t,x) = V\left(\sum_{j=0}^{1}\lambda_j(t_{i+j},x)\right) = \sum_{j=0}^{1}\lambda_j\frac{|x|}{x^*}V(t_{i+j},\pm x^*) \geq \sum_{j=0}^{1}\lambda_j\frac{|x|}{x^*}x^* = |x|.$$

To prove 3., fix a point $(t,x) \in R_0$ and $(i,\pm)$ with $i = 0, 1 \ldots, n-1$ such that (t,x) lies in $[t_i, t_{i+1}] \times [0, x^*]$ for $(i,+)$ or $[t_i, t_{i+1}] \times [-x^*, 0]$ for $(i,-)$, respectively. Let again $(t,x)=\sum_{j=0}^1 \lambda_j(t_{i+j},x)$ for $\lambda_0,\lambda_1\in[0,1]$ with $\sum_{j=0}^1\lambda_j=1$. We will show $\dot V\big|_{(i,\pm)}(t,x)=V_t(t,x)+V_x(t,x)f(t,x)\le -|x|$, where $V\big|_{(i,\pm)}$ denotes the restriction of V to the rectangle $[t_i,t_{i+1}]\times[0,x^*]$ or $[t_i,t_{i+1}]\times[-x^*,0]$, respectively. By writing $f(t,x)=xg(t,x)$, we obtain with Lemma 3.5, where $\pm$ denotes the rectangle and the sign of x,

$$\begin{aligned}
\dot V(t,x) &= V_t(t,x)+V_x(t,x)xg(t,x)\\
&= \frac{|x|}{x^*}\frac{V_{(t_{i+1},\pm x^*)}-V_{(t_i,\pm x^*)}}{t_{i+1}-t_i}+xg(t,x)\left(\frac{\pm 1}{x^*}\sum_{j=0}^1\lambda_j V_{(t_{i+j},\pm x^*)}\right)\\
&= \frac{|x|}{x^*}\left[\frac{V_{(t_{i+1},\pm x^*)}-V_{(t_i,\pm x^*)}}{t_{i+1}-t_i}+g(t,x)\sum_{j=0}^1\lambda_j V_{(t_{i+j},\pm x^*)}\right]\\
&= \frac{|x|}{x^*}F(t,x)
\end{aligned}$$

using λ_0 and λ_1 from the proof of Lemma 3.5, where we define

$$\begin{aligned}
F(t,x) &= \frac{V_{(t_{i+1},\pm x^*)}-V_{(t_i,\pm x^*)}}{t_{i+1}-t_i}\\
&\quad +g(t,x)\frac{t(V_{(t_{i+1},\pm x^*)}-V_{(t_i,\pm x^*)})+t_{i+1}V(t_i,\pm x^*)-t_iV(t_{i+1},\pm x^*)}{t_{i+1}-t_i}.
\end{aligned}$$

We estimate $|F(\sum_{j=0}^1\lambda_j t_{i+j},x)-\sum_{j=0}^1\lambda_jF(t_{i+j},x)|$ using Taylor expansion of F with respect to t around (t_i,x): there are $t_\alpha,t_{\beta_0},t_{\beta_1}\in(t_i,t_{i+1})$ such that

$$\begin{aligned}
&\left|F\left(\sum_{j=0}^1\lambda_j t_{i+j},x\right)-\sum_{j=0}^1\lambda_jF(t_{i+j},x)\right|\\
&= \Bigg|F(t_i,x)+F_t(t_i,x)\sum_{j=0}^1\lambda_j(t_{i+j}-t_i)+\frac12F_{tt}(t_\alpha,x)\left(\sum_{j=0}^1\lambda_j(t_{i+j}-t_i)\right)^2\\
&\quad -\sum_{j=0}^1\lambda_jF(t_i,x)-\sum_{j=0}^1\lambda_jF_t(t_i,x)(t_{i+j}-t_i)-\frac12\sum_{j=0}^1\lambda_jF_{tt}(t_{\beta_j},x)(t_{i+j}-t_i)^2\Bigg|\\
&\le \frac12\left(|F_{tt}(t_\alpha,x)|+|F_{tt}(t_{\beta_1},x)|\right)(t_{i+1}-t_i)^2\\
&\le M_i(t_{i+1}-t_i)^2, \qquad (6)
\end{aligned}$$

where we define $M_i=\max_{t\in[t_i,t_{i+1}]}\max_{x\in[-x^*,x^*]}|F_{tt}(t,x)|$; note that F_{tt} is continuous. We also have used $\lambda_j\in[0,1]$. Next, we establish a bound for M_i by considering $F_{tt}(t,x)$ for $(t,x)\in[t_i,t_{i+1}]\times[-x^*,x^*]$

$$\begin{aligned}
|F_{tt}(t,x)| &\le \Bigg|g_{tt}(t,x)\frac{t(V_{(t_{i+1},\pm x^*)}-V_{(t_i,\pm x^*)})+t_{i+1}V_{(t_i,\pm x^*)}-t_iV_{(t_{i+1},\pm x^*)}}{t_{i+1}-t_i}\\
&\qquad +2g_t(t,x)\frac{V_{(t_{i+1},\pm x^*)}-V_{(t_i,\pm x^*)}}{t_{i+1}-t_i}\Bigg|\\
&\le |g_{tt}(t,x)|\left(\frac{V_{(t_{i+1},\pm x^*)}(t-t_i)}{t_{i+1}-t_i}+\frac{V_{(t_i,\pm x^*)}(t_{i+1}-t)}{t_{i+1}-t_i}\right)
\end{aligned}$$

$$+ |2g_t(t,x)| \left| \frac{V_{(t_{i+1},\pm x^*)} - V_{(t_i,\pm x^*)}}{t_{i+1}-t_i} \right|$$

$$\leq |g_{tt}(t,x)| \left(\frac{C(t-t_i)}{t_{i+1}-t_i} + \frac{C(t_{i+1}-t)}{t_{i+1}-t_i} \right) + 2|g_t(t,x)|D$$

using Constraints 3.2, 2. and 4.

$$= |g_{tt}(t,x)| \frac{C(t_{i+1}-t_i)}{t_{i+1}-t_i} + 2|g_t(t,x)|D$$

$$= |g_{tt}(t,x)|C + 2|g_t(t,x)|D.$$

Using the definition of $\dot{g}_{\max}$ and $\ddot{g}_{\max}$ in Constraints 3.2 we have

$$M_i \quad \leq \quad C\ddot{g}_{\max} + 2D\dot{g}_{\max}. \tag{7}$$

Now we prove that $\dot{V}(x,t) = \frac{|x|}{x^*}F(t,x) \leq -|x|$ for V restricted to $[t_i, t_{i+1}] \times [0, x^*]$ if $x \geq 0$ or $[t_i, t_{i+1}] \times [-x^*, 0]$ if $x \leq 0$. We have

$$\dot{V}\left(\sum_{j=0}^{1} \lambda_j (t_{i+j}, x)\right)$$

$$= \quad \frac{|x|}{x^*} F\left(\sum_{j=0}^{1} \lambda_j t_{i+j}, x\right)$$

$$= \quad \frac{|x|}{x^*} \left[\sum_{j=0}^{1} \lambda_j F(t_{i+j}, x) + F\left(\sum_{j=0}^{1} \lambda_j t_{i+j}, x\right) - \sum_{j=0}^{1} \lambda_j F(t_{i+j}, x)\right]$$

$$\leq \quad \frac{|x|}{x^*} \left[\sum_{j=0}^{1} \lambda_j F(t_{i+j}, x) + M_i (t_{i+1}-t_i)^2\right] \quad \text{using (6)}$$

$$\leq \quad \frac{|x|}{x^*} \left[\lambda_0 F(t_i, x) + \lambda_1 F(t_{i+1}, x) + (t_{i+1}-t_i)^2 (C\ddot{g}_{\max} + 2D\dot{g}_{\max})\right] \text{ using (7)}$$

$$= \quad \frac{|x|}{x^*} \Bigg[\lambda_0 \frac{V_{(t_{i+1},\pm x^*)} - V_{(t_i,\pm x^*)}}{t_{i+1}-t_i} + \lambda_0 g(t_i, x) V_{(t_i,\pm x^*)} + \lambda_1 \frac{V_{(t_{i+1},\pm x^*)} - V_{(t_i,\pm x^*)}}{t_{i+1}-t_i}$$

$$+\lambda_1 g(t_{i+1}, x) V_{(t_{i+1},\pm x^*)} + (t_{i+1}-t_i)^2 (C\ddot{g}_{\max} + 2D\dot{g}_{\max})\Bigg]$$

$$\leq \quad \frac{|x|}{x^*} \left[\lambda_0(-x^* - E_i) + \lambda_1(-x^* - E_i) + (t_{i+1}-t_i)^2 (C\ddot{g}_{\max} + 2D\dot{g}_{\max})\right]$$

using Constraints 3.2, 5.

$$= \quad \frac{|x|}{x^*} \left[(\lambda_0 + \lambda_1)(-x^* - E_i) + (t_{i+1}-t_i)^2 (C\ddot{g}_{\max} + 2D\dot{g}_{\max})\right]$$

$$= \quad \frac{|x|}{x^*} \left[-x^* - E_i + (t_{i+1}-t_i)^2 (C\ddot{g}_{\max} + 2D\dot{g}_{\max})\right]$$

$$= \quad \frac{|x|}{x^*}(-x^*) = -|x|$$

where we have used the definition of E_i. By Remark 3.6 it follows that V is a Lyapunov function. This proves the theorem. □

This finishes this section. We will now consider the area $D \setminus R_0^\circ$, the domain outside of the close neighbourhood of $x = 0$.

3.2. **Lyapunov Function in $D \setminus R_0^\circ$.** We now consider the area $D \setminus R_0^\circ$, which we will triangulate and where we will use the classical CPA functions. At the boundary ∂R_0, we use the same vertices $(t_i, \pm x^*)$ as in R_0, which will ensure that the function overall is continuous. The simplest way of achieving this is to use the standard triangulation sketched in Figure 3 outside R_0, but we will prove the results using a more general triangulation.

Definition 3.7. Let $p \in \{0, 1, 2\}$. A p-simplex is a set

$$S_\nu = \mathrm{co}(\tilde{x}_0, \ldots, \tilde{x}_p) := \{(t, x) \in S_T \times \mathbb{R} \colon (t, x) = \sum_{i=0}^{p} \lambda_i \tilde{x}_i, \sum_{i=0}^{p} \lambda_i = 1, \lambda_i \geq 0\}$$

with vertices $\tilde{x}_i \in S_T \times \mathbb{R}$, such that $\tilde{x}_0, \ldots, \tilde{x}_p$ are affinely independent (i.e. the vectors $\tilde{x}_1 - \tilde{x}_0, \ldots, \tilde{x}_p - \tilde{x}_0$ are linearly independent) – this definition does not depend on the choice of $\tilde{x}_0$.

We require that the triangulation of $D \setminus R_0^\circ$ is compatible at the boundary with the triangulation of $\partial R_0 = S_T \times \{\pm x^*\}$, namely that its vertices in ∂R_0 are the $(t_i, \pm x^*)$ from Constraints 3.2.

Definition 3.8. A triangulation of $D \setminus R_0^\circ$ is a collection of finitely many 2-simplices $S_\nu = \mathrm{co}(\tilde{x}_0^\nu, \tilde{x}_1^\nu, \tilde{x}_2^\nu)$, $\nu = 1, \ldots, N$, with vertices $\mathcal{V}_\nu = \{\tilde{x}_0^\nu, \tilde{x}_1^\nu, \tilde{x}_2^\nu\} \subset S_T \times \mathbb{R}$, such that for any two simplices S_ν and S_μ with $\nu \neq \mu$ the intersection $S_\nu \cap S_\mu$ is either empty, or a p-simplex, where $p \in \{0, 1\}$ and $D \setminus R_0^\circ = \bigcup_{\nu=1}^{N} S_\nu$. Denote $\mathcal{V} = \bigcup_\nu \mathcal{V}_\nu$.

Moreover, we require that for each ν, if $x_i^\nu \geq x^*$ for $i \in \{0, 1, 2\}$, then also $x_j^\nu \geq x^*$ holds for all $j \in \{0, 1, 2\}$, while if $x_i^\nu \leq -x^*$ $i \in \{0, 1, 2\}$, then also $x_j^\nu \leq -x^*$ holds for all $j \in \{0, 1, 2\}$.

We say that the triangulation is compatible with the triangulation of ∂R_0 from Constraints 3.2 if the following holds: If $S_\nu \cap \partial R_0 \neq \emptyset$, then $\mathcal{V}_\nu \cap \partial R_0 = \{(t_i, \pm x^*)\}$ with $i \in \{0, \ldots, n\}$ or $\mathcal{V}_\nu \cap \partial R_0 = \{(t_i, \pm x^*), (t_{i+1}, \pm x^*)\}$ with $i \in \{0, \ldots, n-1\}$.

Finally, we denote

$$h_\nu := \max_{i,j=0,1,2} \mathrm{dist}(\tilde{x}_i, \tilde{x}_j),$$

where $\mathrm{dist}((s, x), (t, y))^2 = |x - y|^2 + \min(|s - t|, |s - t + T|, |s - t - T|)^2$ denotes the distance in $S_T \times \mathbb{R}$. The triangulation is called (h, d)-bounded, if

1. $h_\nu \leq h$ and
2. $h_\nu \|X_\nu^{-1}\|_2 \leq d$

holds for all $\nu = 1, \ldots, N$, where $X_\nu = \begin{pmatrix} t_1^\nu - t_0^\nu & x_1^\nu - x_0^\nu \\ t_2^\nu - t_0^\nu & x_2^\nu - x_0^\nu \end{pmatrix}$ denotes the shape matrix of the simplex $S_\nu = \mathrm{co}((t_0^\nu, x_0^\nu), (t_1^\nu, x_1^\nu), (t_2^\nu, x_2^\nu))$, see [11, Definition 2.8].

Remark 3.9. The standard triangulation is $(h, 2\sqrt{2})$ bounded for $h > \sqrt{2}$, see [11, Lemma 2.15]. The scaled standard triangulation, multiplied by s is $(sh, 2\sqrt{2})$ bounded for $h > \sqrt{2}$, so we can use $d = 2\sqrt{2}$ for any s.

We can now define the CPA function V on this triangulation. We will show that if we extend the definition of V on R_0 as in Definition 3.3 to the entire set D, then the resulting function is continuous on D.

Definition 3.10. Consider the triangulation S_ν, $\nu \in \{1, \ldots, N\}$ of $D \setminus R_0^\circ$ as in Definition 3.8. Assume that for each vertex $\tilde{x} = (t, x) \in \mathcal{V}$, the value $V_{(t,x)}$ is given. For $(t, x) \in R_0$, V is defined in Definition 3.3.

Let $(t,x) \in D \setminus R_0^\circ$. Then there exists $\nu \in \{1,\dots,N\}$ such that $(t,x) \in S_\nu = \mathrm{co}(\tilde{x}_0^\nu, \tilde{x}_1^\nu, \tilde{x}_2^\nu)$ and hence there exist $\lambda_0, \lambda_2, \lambda_2 \in [0,1]$ such that $\sum_{i=0}^2 \lambda_i = 1$ and $(t,x) = \sum_{i=0}^2 \lambda_i \tilde{x}_i^\nu$. Then, we define

$$V(t,x) = \sum_{i=0}^{2} \lambda_i V_{\tilde{x}_i}. \tag{8}$$

Lemma 3.11. *The function $V\colon D \to \mathbb{R}$ defined above is continuous in D and $V\big|_{S_\nu}$ is given by*

$$V(t,x) = \nabla V_\nu \cdot \begin{pmatrix} t \\ x \end{pmatrix} + a_\nu, \tag{9}$$

where $a_\nu \in \mathbb{R}$ and $\nabla V_\nu = X_\nu^{-1} \begin{pmatrix} V_{\tilde{x}_1} - V_{\tilde{x}_0} \\ V_{\tilde{x}_2} - V_{\tilde{x}_0} \end{pmatrix}$ is the (constant) gradient of $V\big|_{S_\nu}$; the shape matrix X_ν was defined in Definition 3.8.

Proof. To prove this lemma it is sufficient to prove the continuity of $V(t,x)$ on $S_T \times \{\pm x^*\}$.

First note that both Definition 3.3 and Definition 3.10 define the function V at ∂R_0. We will show below that these values are the same and thus the function is well defined.

Fix a general point $(t, \pm x^*)$ for $t \in S_T$. Let us first consider R_0 and let $i \in \{0,\dots,n-1\}$ be such that $t \in [t_i, t_{i+1})$. Then the Lyapunov function in R_0 for this point by Definition 3.3 is given by

$$V(t, \pm x^*) = \frac{|x^*|}{x^*} \sum_{j=0}^{1} \lambda_j V_{(t_{i+j}, \pm x^*)} = \lambda_0 V_{(t_i, \pm x^*)} + \lambda_1 V_{(t_{i+1}, \pm x^*)}, \tag{10}$$

where $t = \sum_{j=0}^1 \lambda_j t_{i+j}$ with $\lambda_j \geq 0$ and $\sum_{j=0}^1 \lambda_j = 1$.

Since $(t, \pm x^*)$ also lies in $D \setminus R_0^\circ$, it lies in a triangle $(t, \pm x^*) \in \mathrm{co}(\tilde{x}_0^\nu, \tilde{x}_1^\nu, \tilde{x}_2^\nu)$. Either one or two of the vertices lie on the line $S_T \times \{\pm x^*\}$; all three is contradictory to them being linearly affine. Indeed, if none of them lies on the line, say, x^*, then $x_j^\nu > x^*$ for all j and any point $(t,x) \in \mathrm{co}(\tilde{x}_0^\nu, \tilde{x}_1^\nu, \tilde{x}_2^\nu)$ satisfies

$$x = \sum_{j=0}^{2} \lambda_j x_j > \sum_{j=0}^{2} \lambda_j x^* = x^*,$$

which is a contradiction to $(t, x^*) \in S_\nu$.

Now, if exactly one vertex is on the line, say $x_0^\nu = x^*$ and $x_j^\nu > x^*$ for $j = 1, 2$, then, with a similar argumentation as above, we have that the point $(t, x^*) = (t_0^\nu, x_0^\nu)$, $\lambda_0 = 1$ and the value of V defined by Definition 3.10 is

$$V(t,x) = V_{(t_0, x_0)}$$

and the same as in (10).

If exactly two vertices are on the line, say $x_j^\nu = x^*$ for $j = 0, 1$ and $x_2^\nu > x^*$, then, with a similar argumentation as above, we have that $(t, x^*) = \sum_{j=0}^2 (t_j^\nu, x_j^\nu)$, $\lambda_2 = 0$. By Definition 3.8 we have $t_0^\nu = t_i$ and $t_1^\nu = t_{i+1}$ and the value of V defined by Definition 3.10 is

$$V(t,x) = \sum_{j=0}^{1} \lambda_j V_{(t_j^\nu, x_j^\nu)} = \lambda_0 V_{(t_i, x^*)} + \lambda_1 V_{(t_{i+1}, x^*)}$$

and the same as in (10). □

Now we can formulate the main result of this section. Theorem 3.13 shows the existence of a CPA Lyapunov function for the system $\dot{x} = f(t,x)$ on D if Constraints 3.2 and 3.12 hold.

To show that V satisfies the same estimates in $D \setminus R_0$ as in R_0, we require the following constraints on the values $V_{(t,x)}$ for vertices (t,x); note that these are the classical constraints, used for CPA functions before.

Constraints 3.12. Let $\{S_\nu\}_{\nu=1,\dots,N}$ be a triangulation of $D\setminus R_0^\circ$ and denote $H_f = \max_{(t,x)\in D} \|\mathrm{Hess}\, f(t,x)\|_2$. The constraints on the values of $V_{(t,x)}$ for each vertex (t,x) of the triangulation and for the values $D_\nu \in \mathbb{R}$ are as follows:

1. $V_{(t,x)} \geq |x|$ for all $(t,x) \in \mathcal{V}$,
2. $\|\nabla V_\nu\|_1 \leq D_\nu$ for all ν,
3. for all ν and all $i = 0,1,2$:

$$\nabla V_\nu \cdot \begin{pmatrix} 1 \\ f(t_i^\nu, x_i^\nu) \end{pmatrix} + D_\nu H_f h_\nu^2 \leq -|x_i^\nu|,$$

where $S_\nu = \mathrm{co}(\tilde{x}_0^\nu, \tilde{x}_1^\nu, \tilde{x}_2^\nu) = \mathrm{co}((t_0^\nu, x_0^\nu), (t_1^\nu, x_1^\nu), (t_2^\nu, x_2^\nu))$ and $\nabla = \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix}$.

Theorem 3.13. *Consider the periodic system $\dot{x} = f(t,x)$, for $f \in C^3(S_T \times \mathbb{R}, \mathbb{R})$ and $f(t,0) = 0$ for all t. Let D be a compact, connected set such that $S_T \times \{0\} \subset D^\circ$. Let $x^* > 0$ be so small that $R_0 := S_T \times [-x^*, x^*] \subset D^\circ$ and let $0 = t_0 < t_1 < \ldots < t_n = T$ be given. Let $\{S_\nu\}_{\nu=1,\dots,N}$ be a triangulation of $D \setminus R_0^\circ$ as in Definition 3.8, compatible with the triangulation of ∂R_0, and let the values $V_{(t,x)}$ for all $(t,x) \in \mathcal{V}$ satisfy Constraints 3.2 and 3.12.*

Then, the function $V\colon D \to \mathbb{R}$, defined on R_0 by Definition 3.3 and on $D \setminus R_0^\circ$ by Definition 3.10 is continuous and a Lyapunov function of this system satisfying

1. *$V(t,0) = 0$ for all $t \in S_T$,*
2. *$V(t,x) \geq |x|$ for all $(t,x) \in D$,*
3. *$D_+V(t,x) \leq -|x|$ for all $(t,x) \in D^\circ$.*

Proof. We have already shown that V is continuous and we also have verified the properties for $(t,x) \in R_0^\circ$.

Now consider a point $(t,x) \in D \setminus R_0^\circ$; note that $x \neq 0$, so we do not need to show the first property. We assume $(t,x) \in S_\nu$ and can thus write $(t,x) = \sum_{i=0}^2 \lambda_i \tilde{x}_i^\nu$ with $\lambda_i \in [0,1]$ and $\sum_{i=0}^2 \lambda_i = 1$.

For the second property we have, using 1. of Constraints 3.12

$$V(t,x) \quad = \quad V\left(\sum_{i=0}^2 \lambda_i \tilde{x}_i^\nu\right) = \sum_{i=0}^2 \lambda_i V_{(t_i^\nu, x_i^\nu)} \geq \sum_{i=0}^2 \lambda_i |x_i^\nu| \geq \left|\sum_{i=0}^2 \lambda_i x_i^\nu\right| = |x|$$

To prove the third property, we will show for each ν and for all $(t,x) \in S_\nu$ that $\nabla V_\nu \cdot \begin{pmatrix} 1 \\ f(t,x) \end{pmatrix} \leq -|x|$, which implies the result for the Dini derivative, together with point 3. of Theorem 3.4, see Remark 3.6. We also use the following result from [6, Proposition 2.2]

$$\left| f\left(\sum_{j=0}^2 \lambda_j \tilde{x}_j^\nu\right) - \sum_{j=0}^2 \lambda_j f(\tilde{x}_j^\nu) \right| \quad \leq \quad h_\nu^2 H_f^\nu, \tag{11}$$

where $S_\nu = \mathrm{co}(\tilde{x}_0^\nu, \tilde{x}_1^\nu, \tilde{x}_2^\nu)$ is a 2-simplex with diameter h_ν and we denote $H_f^\nu = \max_{(t,x)\in S_\nu} \|\mathrm{Hess}\, f(t,x)\|_2$. Note that by definition $H_f^\nu \le H_f$ for all ν. We have

$$\begin{aligned}
&\nabla V_\nu \cdot \begin{pmatrix} 1 \\ f(t,x) \end{pmatrix} \\
&= \nabla V_\nu \cdot \begin{pmatrix} 1 \\ f(\sum_{j=0}^2 \lambda_j \tilde{x}_j^\nu) \end{pmatrix} \\
&= \sum_{j=0}^2 \lambda_j \nabla V_\nu \begin{pmatrix} 1 \\ f(\tilde{x}_j^\nu) \end{pmatrix} + \nabla V_\nu \cdot \begin{pmatrix} 1 \\ f(\sum_{j=0}^2 \lambda_j \tilde{x}_j^\nu) \end{pmatrix} - \sum_{j=0}^2 \lambda_j \nabla V_\nu \begin{pmatrix} 1 \\ f(\tilde{x}_j^\nu) \end{pmatrix} \\
&\le \sum_{j=0}^2 \lambda_j \nabla V_\nu \begin{pmatrix} 1 \\ f(\tilde{x}_j^\nu) \end{pmatrix} + \|\nabla V_\nu\|_1 \cdot \left\| \begin{pmatrix} 1 \\ f(\sum_{j=0}^2 \lambda_j \tilde{x}_j^\nu) \end{pmatrix} - \begin{pmatrix} 1 \\ \sum_{j=0}^2 \lambda_j f(\tilde{x}_j^\nu) \end{pmatrix} \right\|_\infty \\
&\le \sum_{j=0}^2 \lambda_j \nabla V_\nu \begin{pmatrix} 1 \\ f(\tilde{x}_j^\nu) \end{pmatrix} + D_\nu \left| f\left(\sum_{j=0}^2 \lambda_j \tilde{x}_j^\nu \right) - \sum_{j=0}^2 \lambda_j f(\tilde{x}_j^\nu) \right| \\
&\qquad\qquad \text{using Constraints 3.12, 2.} \\
&\le \sum_{j=0}^2 \lambda_j \nabla V_\nu \begin{pmatrix} 1 \\ f(\tilde{x}_j^\nu) \end{pmatrix} + D_\nu H_f h_\nu^2 \quad \text{using (11)} \\
&\le \sum_{j=0}^2 \lambda_j \left[\nabla V_\nu \begin{pmatrix} 1 \\ f(\tilde{x}_j^\nu) \end{pmatrix} + D_\nu H_f h_\nu^2 \right] \\
&\le \sum_{j=0}^2 \lambda_j (-|x_j^\nu|) \quad \text{using Constraints 3.12, 3.} \\
&= -\sum_{j=0}^2 \lambda_j |x_j^\nu| = -\left| \sum_{j=0}^2 \lambda_j x_j^\nu \right| = -|x|
\end{aligned}$$

since all x_j^ν have the same sign. This concludes the proof. □

In this section, we have established sufficient conditions for the existence of a CPA Lyapunov function for a periodic orbit. In the next section we will show that, for sufficiently fine triangulations, it is always possible to satisfy these constraints.

4. **Converse theorem.** In this section we will show that for sufficiently fine triangulations it is always possible to find values which satisfy the Constraints 3.2 and 3.12. The main idea is to first show the existence of a Lyapunov function w with certain properties and then to use its values at the vertices of a sufficiently fine triangulation. The function w will be defined as the square root of a smooth Lyapunov function v. While the smooth Lyapunov function v is taken nearly directly from [18, Theorem 4.14], the idea of using $w = \sqrt{v}$ is inspired by [8, Theorems 6 and 7], where a local version of the following theorem is proved; we, however, require a Lyapunov function which is defined on any given compact subset D of the basin of attraction.

Theorem 4.1. *Consider $\dot{x} = f(t,x)$ with $f \in C^3(S_T \times \mathbb{R}, \mathbb{R})$ and assume that $f(t,0) = 0$ for all $t \in S_T$. Furthermore, assume that the zero solution is an exponentially stable solution with basin of attraction $A(0)$. Let $S_T \times \{0\} \subset D^\circ \subset D \subset A(0)$ be a compact set in $S_T \times \mathbb{R}$.*

Then there exists a function $v \in C^3(D, \mathbb{R})$ and constants $c_1, c_2, c_3, c_4 > 0$ satisfying

1. $c_1|x|^2 \le v(t,x) \le c_2|x|^2$
2. $\dot{v}(t,x) \le -c_3|x|^2$
3. $|v_x(t,x)| \le c_4|x|$

for all $(t,x) \in D$. Here, $\dot{v}(t,x) = v_t(t,x) + v_x(t,x)f(t,x)$ denotes the orbital derivative.

For the function $w = \sqrt{v}$ we have $w \in C^3(D \setminus (S_T \times \{0\}), \mathbb{R}) \cap C(D, \mathbb{R})$ and constants $a, b, c, C_{wx} > 0$ satisfying

1. $a|x| \le w(t,x) \le b|x|$ *for all* $(t,x) \in D$,
2. $\dot{w}(t,x) \le -c|x|$ *for all* $(t,x) \in D$,
3. $|w_x(t,x)| \le C_{wx}$ *for all* $(t,x) \in D$ *with* $x \neq 0$.

Proof. In a similar way as [18, Theorem 4.14.], but for a time-periodic system on a compact subset D of its basin of attraction, we can define a function $v \in C^3(D, \mathbb{R})$ satisfying the properties mentioned in the theorem. Note that the conditions for exponential stability hold in the entire compact set D. The function is of the form $v(t,x) = \int_t^{t+\delta} \phi^x(\tau; t, x)^2 \, d\tau$ for suitable $\delta > 0$. Note that $v(t,x) = 0$ if and only if $x = 0$, and $v(t,x) > 0$ for $x \neq 0$.

We define $w(t,x) = \sqrt{v(t,x)}$, which is continuous on D. For the derivatives we have

$$w_t(t,x) = \frac{v_t(t,x)}{2w(t,x)}, \quad w_x(t,x) = \frac{v_x(t,x)}{2w(t,x)}, \quad \text{and} \quad \dot{w}(t,x) = \frac{\dot{v}(t,x)}{2w(t,x)}$$

for $x \neq 0$. Note that $\dot{w}$ can still be defined on $x = 0$ as the orbital derivative $\dot{w}(t,x) = \frac{d}{d\tau} w(\phi(\tau; t, x))\big|_{\tau = t}$ exists and is 0 for $x = 0$. We have

$$\sqrt{c_1}|x| \quad \le \quad \sqrt{v(t,x)} = w(t,x) \quad \le \quad \sqrt{c_2}|x|$$

which shows 1. with $a = \sqrt{c_1}$ and $b = \sqrt{c_2}$. For $x \neq 0$ we have

$$\dot{w}(t,x) \le -c_3 \frac{|x|^2}{2w(t,x)} \le -\frac{c_3}{2\sqrt{c_2}} \frac{|x|^2}{|x|} = -c|x|$$

and

$$|w_x(t,x)| \le \frac{c_4|x|}{2w(t,x)} \le \frac{c_4}{2\sqrt{c_1}} =: C_{wx}$$

which shows the theorem with $c = \frac{c_3}{2\sqrt{c_2}}$. □

Remark 4.2. By replacing the functions v and w with the functions $\tilde{v} = \tilde{c}^2 v$ and $\tilde{w} = \tilde{c}\sqrt{v}$, respectively, with $\tilde{c} = \max\left(\frac{1}{\sqrt{c_1}}, \frac{4\sqrt{c_2}}{c_3}\right)$, we can achieve that $a \ge 1$ and $c \ge 2$ in Theorem 4.1; we will assume this in the next theorem.

We are now ready to prove a converse theorem, ensuring that the Constraints 3.2 and 3.12 can be fulfilled, if the triangulation is sufficiently fine.

Theorem 4.3. *Consider the system $\dot{x} = f(t,x)$ with $f \in C^3(S_T \times \mathbb{R}, \mathbb{R})$ and $f(t,0) = 0$ for all t. Let the zero solution be exponentially stable with basin of attraction $A(0)$, let D be a compact subset of $S_T \times \mathbb{R}$ with $S_T \times \{0\} \subset D^\circ \subset D \subset A(0)$ and fix $d > 0$.*

Then there exists a constant $\overline{x} > 0$ with the following property: For any $x^* \in (0, \overline{x}]$ there exists a constant $\overline{h} = \overline{h}(x^*) > 0$ such that for $R_0 = S_T \times [-x^*, x^*]$ and any $0 = t_0 < t_1 < \ldots < t_n = T$ with $\max_{i=0,\ldots,n-1} |t_{i+1} - t_i| \leq \overline{h}$ and any $(\overline{h}, d)$-bounded triangulation of $D \setminus R_0^\circ$, which is compatible to the triangulation of ∂R_0, there are values for the $V_{(t,x)}$, $(t,x) \in \mathcal{V}$ and for D_ν, C, D, such that Constraints 3.2 and 3.12 are satisfied.

Proof. Denote by $v, w\colon D \to \mathbb{R}$ the functions from Theorem 4.1 with $a \geq 1$ and $c \geq 2$, see Remark 4.2, and by $g \in C^2(S_T \times \mathbb{R}, \mathbb{R})$ the function from Lemma 3.1. Fix $\delta > 0$ such that $S_T \times [-\delta, \delta] \subset D^\circ$. We define the following constants, noting that the functions are continuous on the compact sets

$$\begin{aligned}
C_g &= \max_{(t,x)\in S_T\times[-\delta,\delta]} |g(t,x)|,\\
C_{gx} &= \max_{(t,x)\in S_T\times[-\delta,\delta]} |g_x(t,x)|,\\
C_{gt} &= \max_{(t,x)\in S_T\times[-\delta,\delta]} |g_t(t,x)|,\\
C_{gtt} &= \max_{(t,x)\in S_T\times[-\delta,\delta]} |g_{tt}(t,x)|,\\
C_{vxx} &= \max_{(t,x)\in S_T\times[-\delta,\delta]} |v_{xx}(t,x)|,\\
C_{vxxx} &= \max_{(t,x)\in S_T\times[-\delta,\delta]} |v_{xxx}(t,x)|,\\
H_f &= \max_{(t,x)\in D} \|\operatorname{Hess} f(t,x)\|_2,\\
F &= \max\left(1, \max_{(t,x)\in D} |f(t,x)|\right).
\end{aligned}$$

Using the above constants as well as those defined in Theorem 4.1, we set:

$$\begin{aligned}
\overline{x} &= \min\left(\delta, \frac{c}{4\left(\frac{5}{12a} C_{vxxx} C_g + C_{wx} C_{gx}\right)}\right),\\
\overline{C} &= b\overline{x}.
\end{aligned}$$

Now fix $x^* \in (0, \overline{x}]$, $R_0 = S_T \times [-x^*, x^*]$ and define

$$C_{wt} = \max_{(t,x)\in S_T\times\{\pm x^*\}} |w_t(t,x)|, \tag{12}$$

$$C_{wtt} = \max_{(t,x)\in S_T\times\{\pm x^*\}} |w_{tt}(t,x)|, \tag{13}$$

$$\overline{D} = \frac{C_{wtt}}{2} + C_{wt}, \tag{14}$$

$$H_w = \max_{(t,x)\in D\setminus R_0^\circ} \|\operatorname{Hess} w(t,x)\|_2, \tag{15}$$

$$D' = \sqrt{2} d H_w + \max_{(t,x)\in D\setminus R_0^\circ} \|\nabla w(t,x)\|_1, \tag{16}$$

$$\overline{h} = \min\left(1, \frac{cx^*}{4\left(\frac{C_{wtt}}{2} + 2C_{gt}\overline{D} + C_{gtt}\overline{C}\right)}, \frac{x^*}{\sqrt{2} d H_w F + D' H_f}\right). \tag{17}$$

Fix $t_0 = 0 < t_1 < \ldots < t_n = T$ with $\max_{i=0,\ldots,n-1} |t_{i+1} - t_i| \leq \overline{h}$. Fix a $(\overline{h}, d)$-bounded triangulation in $D \setminus R_0^\circ$, which is compatible with the triangulation of ∂R_0.

We define the values $V_{(t,x)}$ by

$$V_{(t,x)} = w(t,x) \text{ for all } (t,x) \in \mathcal{V}.$$

Furthermore, set $C = \overline{C}$, and $D = \overline{D}$ in Constraints 3.2 and $D_\nu = D'$ in Constraints 3.12 for all ν. We now show that for these values, Constraints 3.2 and 3.12 are fulfilled.

Constraints 3.2

1. The first constraint follows from the first property of w, namely $w(t,x) \geq a|x|$ for all $(t,x) \in D$. As $a \geq 1$, $V_{(t_i, \pm x^*)} = w(t_i, \pm x^*) \geq x^*$.

2. The second constraint follows from the first property of w, namely $w(t,x) \leq b|x|$ for all $(t,x) \in D$. From this we have

$$V_{(t_i, \pm x^*)} = w(t_i, \pm x^*) \leq bx^* \leq \overline{C} = C, \tag{18}$$

since $x^* \leq \overline{x}$.

3. $V_{(t_0, x^*)} = V_{(t_n, x^*)}$ holds since $w(t,x)$ is T-periodic.

4. By the mean value theorem there exists $t_i < s_i^\pm < t_{i+1}$ such that

$$w(t_{i+1}, \pm x^*) - w(t_i, \pm x^*) = w_t(s_i^\pm, \pm x^*)(t_{i+1} - t_i),$$

i.e.

$$\left| \frac{V_{(t_{i+1}, \pm x^*)} - V_{(t_i, \pm x^*)}}{t_{i+1} - t_i} \right| = |w_t(s_i^\pm, \pm x^*)| \leq C_{wt} \leq \overline{D}.$$

5. We only show

$$\frac{V_{(t_{i+1}, \pm x^*)} - V_{(t_i, \pm x^*)}}{t_{i+1} - t_i} + g_{\max}(i, \pm) V_{(t_i, \pm x^*)} + E_i \leq -x^*;$$

the case with $g_{\max}(i+1, \pm)$ instead of $g_{\max}(i, \pm)$ can be shown analogously. Using property 2. of w, we know that

$$\dot{w}(t_i, \pm x^*) = w_t(t_i, \pm x^*) + w_x(t_i, \pm x^*) f(t_i, \pm x^*) \leq -cx^*.$$

We use $w(t,x) = \sqrt{v(t,x)}$ and the Taylor expansions of v and v_x around $(t_i, 0)$ as well as $v(t,0) = v_x(t,0) = 0$, which follows from $|v_x(t,x)| \leq c_4|x|$. Hence, there are $x_1, x_2 \in (0, x^*)$ (for $+x^*$) or $x_1, x_2 \in (-x^*, 0)$ (for $-x^*$) such that

$$\begin{aligned}
&|w(t_i, \pm x^*) - (\pm x^*) w_x(t_i, \pm x^*)| \\
&= \left| \sqrt{v(t_i, \pm x^*)} - (\pm x^*) \frac{v_x(t_i, \pm x^*)}{2\sqrt{v(t_i, \pm x^*)}} \right| \\
&= \frac{1}{\sqrt{v(t_i, \pm x^*)}} \left| v(t_i, \pm x^*) - \frac{1}{2}(\pm x^*) v_x(t_i, \pm x^*) \right| \\
&\leq \frac{1}{ax^*} \left| v(t_i, 0) \pm v_x(t_i, 0) x^* + \frac{1}{2} v_{xx}(t_i, 0)(x^*)^2 \pm \frac{1}{6} v_{xxx}(t_i, x_1)(x^*)^3 \right. \\
&\qquad \left. - \frac{1}{2}(\pm x^*) \left[v_x(t_i, 0) \pm v_{xx}(t_i, 0) x^* + \frac{1}{2} v_{xxx}(t_i, x_2)(x^*)^2 \right] \right| \\
&= \frac{1}{ax^*} \left| \frac{1}{6} v_{xxx}(t_i, x_1)(\pm x^*)^3 - \frac{1}{4}(\pm x^*)^3 v_{xxx}(t_i, x_2)] \right|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{ax^*}\frac{5}{12}C_{vxxx}(x^*)^3 \\
&= \frac{5}{12a}C_{vxxx}(x^*)^2. \qquad (19)
\end{aligned}
$$

Note, that there exists $x_1 \in [0, x^*]$ such that $g_{\max}(i,+) = g(t_i, x_1)$ and an $x_2 \in (x_1, x^*)$ such that $g(t_i, x^*) - g(t_i, x_1) = g_x(t_i, x_2)(x^* - x_1)$. Hence, we have

$$
|g_{\max}(i,+) - g(t_i, x^*)| \le x^* C_{gx} \qquad (20)
$$

For $-x^*$ we obtain in a similar way

$$
|g_{\max}(i,-) - g(t_i, -x^*)| \le x^* C_{gx}. \qquad (21)
$$

To prove 5., first note that using Taylor expansion there is a $t^* \in (t_i, t_{i+1})$ such that

$$
\frac{w(t_{i+1}, \pm x^*) - w(t_i, \pm x^*)}{t_{i+1} - t_i} = w_t(t_i, \pm x^*) + \frac{t_{i+1} - t_i}{2} w_{tt}(t^*, \pm x^*).
$$

Hence

$$
\left| \frac{w(t_{i+1}, \pm x^*) - w(t_i, \pm x^*)}{t_{i+1} - t_i} - w_t(t_i, \pm x^*) \right| \le \frac{t_{i+1} - t_i}{2} C_{wtt}. \qquad (22)
$$

Using this, the second property of the Lyapunov function w, the definition of $\dot{w}$ and $f(t,x) = xg(t,x)$, we estimate

$$
\begin{aligned}
&\frac{V_{(t_{i+1}, \pm x^*)} - V_{(t_i, \pm x^*)}}{t_{i+1} - t_i} + g_{\max}(i, \pm) V_{(t_i, \pm x^*)} \\
= \;& \dot{w}(t_i, \pm x^*) + \frac{w(t_{i+1}, \pm x^*) - w(t_i, \pm x^*)}{t_{i+1} - t_i} + g_{\max}(i,\pm) w(t_i, \pm x^*) - \dot{w}(t_i, \pm x^*) \\
\le \;& -cx^* + \frac{w(t_{i+1}, \pm x^*) - w(t_i, \pm x^*)}{t_{i+1} - t_i} - w_t(t_i, \pm x^*) \\
& + g_{\max}(i,\pm) w(t_i, \pm x^*) - (\pm x^*) g(t_i, \pm x^*) w_x(t_i, \pm x^*) \\
\le \;& -cx^* + C_{wtt}(t_{i+1} - t_i)/2 \\
& + |g_{\max}(i,\pm) w(t_i, \pm x^*) - g_{\max}(i,\pm)(\pm x^*) w_x(t_i, \pm x^*)| \\
& + |g_{\max}(i,\pm)(\pm x^*) w_x(t_i, \pm x^*) - (\pm x^*) g(t_i, \pm x^*) w_x(t_i, \pm x^*)| \\
= \;& -cx^* + C_{wtt}(t_{i+1} - t_i)/2 + |g_{\max}(i,\pm)|\, |w(t_i, \pm x^*) - (\pm x^*) w_x(t_i, \pm x^*)| \\
& + |x^* w_x(t_i, \pm x^*)|\, |g_{\max}(i,\pm) - g(t_i, \pm x^*))| \\
\le \;& -cx^* + \frac{t_{i+1} - t_i}{2} C_{wtt} + \left(\frac{5}{12a} C_{vxxx} C_g + C_{wx} C_{gx} \right) (x^*)^2
\end{aligned}
$$

by (22), (19), (20) and (21), respectively.

Finally, taking the term $E_i = (t_{i+1} - t_i)^2 (2\dot{g}_{\max} D + \ddot{g}_{\max} C) \le \overline{h}^2 (2C_{gt}\overline{D} + C_{gtt}\overline{C})$ into account and using $t_{i+1} - t_i \le \overline{h} \le 1$, we obtain

$$
\begin{aligned}
&\frac{V_{(t_{i+1}, \pm x^*)} - V_{(t_i, \pm x^*)}}{t_{i+1} - t_i} + g_{\max}(i, \pm) V_{(t_i, \pm x^*)} + E_i \\
\le \;& -cx^* + \overline{h}\frac{C_{wtt}}{2} + \left(\frac{5}{12a} C_{vxxx} C_g + C_{wx} C_{gx} \right) (x^*)^2 + \overline{h}^2 (2C_{gt}\overline{D} + C_{gtt}\overline{C}) \\
\le \;& -cx^* + \overline{h} \left(\frac{C_{wtt}}{2} + 2C_{gt}\overline{D} + C_{gtt}\overline{C} \right) + \overline{x} x^* \left(\frac{5}{12a} C_{vxxx} C_g + C_{wx} C_{gx} \right) \\
\le \;& -cx^* + \frac{cx^*}{4} + \frac{cx^*}{4}
\end{aligned}
$$

$$\begin{aligned} &= -\frac{cx^*}{2} \\ &\leq -x^* \end{aligned}$$

as $c \geq 2$ by definition of $\overline{x}$ and $\overline{h}$.

Now we show

Constraints 3.12

1. We have already shown $V_{(t_i,x_i)} \geq |x_i|$ above.
2. As the triangulation is $(\overline{h}, d)$-bounded, we have

$$h_\nu \|X_\nu^{-1}\|_2 \leq d, \tag{23}$$

for all ν, where $X_\nu = \begin{pmatrix} t_1^\nu - t_0^\nu & x_1^\nu - x_0^\nu \\ t_2^\nu - t_0^\nu & x_2^\nu - x_0^\nu \end{pmatrix}$ denotes the shape matrix of the simplex

$$S_\nu = \text{co}((t_0^\nu, x_0^\nu), (t_1^\nu, x_1^\nu), (t_2^\nu, x_2^\nu)).$$

We have

$$X_\nu \nabla V_\nu = w_\nu = \begin{pmatrix} w(t_1^\nu, x_1^\nu) - w(t_0^\nu, x_0^\nu) \\ w(t_2^\nu, x_2^\nu) - w(t_0^\nu, x_0^\nu) \end{pmatrix},$$

see Lemma 3.11.

Using [9, (19)] we can deduce that

$$\|w_\nu - X_\nu \nabla w(t_i, x_i)\|_1 \leq \frac{1}{2} 2h_\nu^2 H_w. \tag{24}$$

Then, let us consider the difference between ∇V_ν and $\nabla w(t_i, x_i)$.

$$\begin{aligned} \|\nabla V_\nu - \nabla w(t_i, x_i)\|_1 &= \|X_\nu^{-1} w_\nu - \nabla w(t_i, x_i)\|_1 \\ &= \|X_\nu^{-1}\|_1 \|w_\nu - X_\nu \nabla w(t_i, x_i)\|_1 \\ &\leq \sqrt{2}\, \|X_\nu^{-1}\|_2 \|w_\nu - X_\nu \nabla w(t_i, x_i)\|_1 \\ &\leq \sqrt{2} \frac{d}{h_\nu} h_\nu^2 H_w \quad \text{by (23) and (24)} \\ &= \sqrt{2} d h_\nu H_w. \end{aligned} \tag{25}$$

We have for any $i = 0, 1, 2$ by (25)

$$\begin{aligned} \|\nabla V_\nu\|_1 &\leq \|\nabla V_\nu - \nabla w(t_i, x_i)\|_1 + \|\nabla w(t_i, x_i)\|_1 \\ &\leq \sqrt{2} d h_\nu H_w + \max_{(t,x) \in D \setminus R_0^\circ} \|\nabla w(t_i, x_i)\|_1 \leq D' \end{aligned}$$

since $h_\nu \leq \overline{h} \leq 1$, which shows 2. as $D_\nu = D'$.

3. Lastly, we need to prove that $\nabla V_\nu \cdot \begin{pmatrix} 1 \\ f(t_i, x_i) \end{pmatrix} + D_\nu H_f h_\nu^2 \leq -|x_i|$ for all vertices (t_i, x_i) of S_ν. We have, using $h_\nu \leq \overline{h} \leq 1$ $c \geq 2$ and $|x_i| \geq x^*$, as well as $D_\nu = D'$

$$\begin{aligned} \nabla V_\nu \cdot \begin{pmatrix} 1 \\ f(t_i, x_i) \end{pmatrix} + D_\nu H_f h_\nu^2 &\leq \nabla w(t_i, x_i) \cdot \begin{pmatrix} 1 \\ f(t_i, x_i) \end{pmatrix} \\ &\quad + (\nabla V_\nu - \nabla w(t_i, x_i)) \cdot \begin{pmatrix} 1 \\ f(t_i, x_i) \end{pmatrix} + D' H_f h_\nu^2 \\ &\leq -c|x_i| + \sqrt{2} d h_\nu H_w \max(1, |f(t_i, x_i)|) \\ &\quad + D' H_f h_\nu^2 \text{ by (25)} \\ &\leq -|x_i| - x^* + (\sqrt{2} d H_w F + D' H_f) \overline{h} \end{aligned}$$

$$\leq \quad -|x_i|$$

by the definition of $\overline{h}$. □

5. **Examples.** We used our novel method to compute Lyapunov functions for two different systems. The first example is the $T = 2\pi$ periodic system

$$\dot{x} = -(1 - \sin(t))x. \tag{26}$$

Note that this system does not possess a classical CPA Lyapunov function as we showed in the Introduction. We set $x^* = 0.02$ and $t_i = i\Delta t$ with $\Delta t = 2\pi/100$ for $i = 0, 1, \dots, 100$. From $g(t, x) = \sin(t) - 1$ one obtains the formulas

$$\dot{g}_{\max} = \ddot{g}_{\max} = 1 \quad \text{and} \quad g_{\max}(i, +) = g_{\max}(i, -) = \sin(t_i) - 1$$

for the constants in Constraints 3.2. Note that $R_0 = [0, 2\pi] \times [-0.02, 0.02]$. We set $D = [0, 2\pi] \times [-1.2, 1.2]$ and triangulated the area $D \setminus R_0^\circ$ with the triangles

$$\begin{aligned} &\mathrm{co}((i\Delta t, \pm jx^*), ((i+1)\Delta t, \pm jx^*), ((i+1)\Delta t, \pm(j+1)x^*)) \quad \text{and} \\ &\mathrm{co}((i\Delta t, \pm jx^*), (i\Delta t, \pm(j+1)x^*), ((i+1)\Delta t, \pm(j+1)x^*)) \quad \text{for} \\ &\qquad i = 0, 1, \dots, 99 \quad \text{and} \quad j = 1, 2 \dots, 59. \end{aligned}$$

We can bound the Hesse-matrix H of f on D by

$$\|H(t, x)\|_2 = \left\| \begin{pmatrix} -x\sin(t) & \cos(t) \\ \cos(t) & 0 \end{pmatrix} \right\|_2 \leq \left\| \begin{pmatrix} 1.2 & 1 \\ 1 & 0 \end{pmatrix} \right\|_2 \leq 1.8,$$

which gives us the constant $H_f = 1.8$ for Constraints 3.12. With the linear solver Gurobi we obtained a feasible solution to the linear programming problem constructed from Constraints 3.2 and 3.12, which parameterizes a Lyapunov function for the system as in Theorem 3.13. The computed Lyapunov function is shown in Figure 4, left. The right-hand side figure shows several level sets of the Lyapunov function. The connected components of those sublevel sets, which include the zero solution, are subsets of the basin of attraction of the zero solution.

The second example is the $T = 2\pi$ periodic system

$$\dot{x} = (\lambda \sin(t) - 1)x + x^2 \tag{27}$$

with $\lambda = 1/2$ from [12]. We use the same triangulation as for the first example. From $g(t, x) = \lambda \sin(t) - 1 + x$ we get the formulas

$$\dot{g}_{\max} = \ddot{g}_{\max} = \lambda, \quad g_{\max}(i, +) = \lambda \sin(t_i) - 1 + x^*, \quad \text{and} \quad g_{\max}(i, -) = \lambda \sin(t_i) - 1$$

for the constants in Constraints 3.2.

We can bound the Hesse-matrix H of f on D by

$$\|H(t, x)\|_2 = \left\| \begin{pmatrix} -\lambda x\sin(t) & \lambda\cos(t) \\ \lambda\cos(t) & 2 \end{pmatrix} \right\|_2 \leq \left\| \begin{pmatrix} 1.2\lambda & \lambda \\ \lambda & 2 \end{pmatrix} \right\|_2 \leq 2.2,$$

which gives us the constant $H_f = 2.2$ for Constraints 3.12. With the linear solver Gurobi we obtained a feasible solution to the linear programming problem constructed from Constraints 3.2 and 3.12, which parameterizes a Lyapunov function for the system as in Theorem 3.13. The computed Lyapunov function is shown in Figure 5, left. The right-hand side figure shows several level sets of the Lyapunov function. The connected components of those sublevel sets, which include the zero solution, are subsets of the basin of attraction of the zero solution.

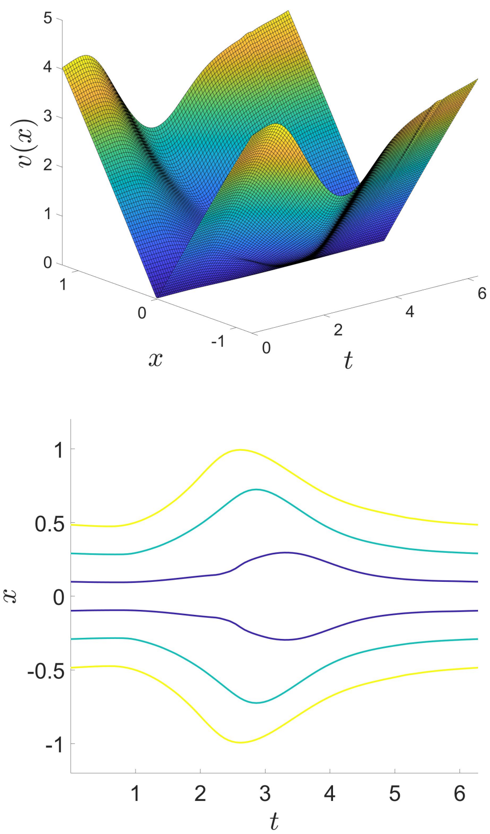

FIGURE 4. Lyapunov function computed for the system (26) $\dot{x} = (\sin(t) - 1)x$ (left) and some of its level-sets (right). Note that each connected component of such a sublevel-set that includes $S_T \times \{0\}$ is a subset of the basin of attraction of the zero solution.

6. **Conclusions.** In this paper we have adapted the numerical construction of CPA Lyapunov functions to time-periodic systems. In particular, we have shown that a modified parameterization of the Lyapunov function in a neighborhood of the periodic orbit is necessary. We introduced a different parameterization and derived linear constraints for the conditions of a Lyapunov function in a neighborhood of the periodic orbit. We have shown that our new method is always able to compute a Lyapunov function, if the periodic orbit is exponentially stable and the triangulation is sufficiently fine. We showed the applicability of our method by computing Lyapunov functions for two different system; for one of them we have shown that it does not possess a CPA Lyapunov function. The paper only considers the case of one spatial dimension, but we are working on the case of dimension $n \geq 2$, which will require a substantially different approach to the triangulation, and, if successful, will be dealt with in a follow-up paper.

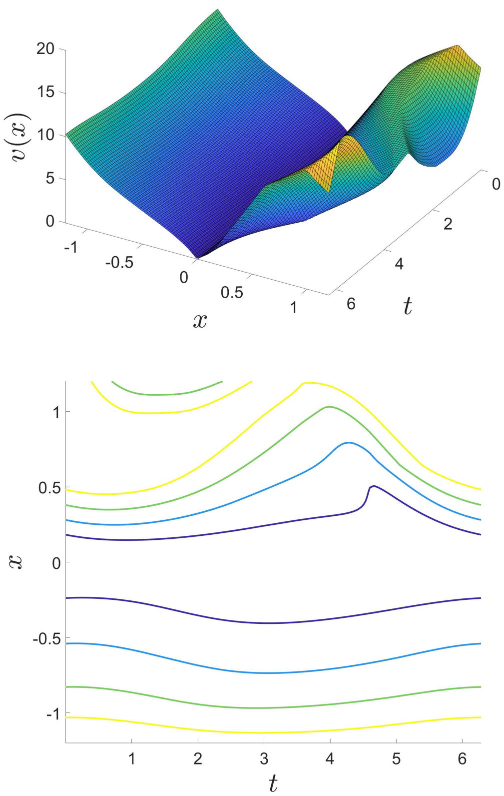

FIGURE 5. Lyapunov function computed for the system (27) $\dot{x} = (\sin(t)/2 - 1)x + x^2$ (left) and some of its level-sets (right). Note that each connected component of such a sublevel-set that includes $S_T \times \{0\}$ is a subset of the basin of attraction of the zero solution.

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Received xxxx 20xx; revised xxxx 20xx; early access xxxx 20xx.