Computation of Lyapunov functions for discrete-time switched linear systems: proof of convergence in the plane ⋆

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Abstract: Recently a novel method was presented to compute convex Lyapunov functions for discrete-time, switched linear systems. The method uses linear programming to parameterize continuous, cone-wise linear, convex Lyapunov functions. We prove that for two-dimensional systems the method is always able to generate a Lyapunov function, whenever the origin is exponentially stable, given that the triangulation used by the algorithm is sufficiently dense.

Keywords: Lyapunov functions, switched systems, discrete-time systems, linear programming.

1. INTRODUCTION

Many systems in science and engineering are most naturally modelled using switching between different dynamical subsystems. Intensive research has been conducted on the stability of switched systems in both continuous- and discrete-time and under different switching rules, see, e.g., Liberzon (2003), Ahmadi and Parrilo (2008), Chesi (2014), Ahmadi and Jungers (2014), Gomide and Lacerda (2018), Hafstein (2018), Hafez and Broucke (2022). Analytical methods for stability analysis are usually difficult or impossible to apply, even in the non-switched case. Therefore, one often resorts to numerical methods, in particular, methods to compute Lyapunov functions that characterize the stability of equilibria, to analyse stability in dynamical systems, see, e.g., Feng (2002), Kalies et al. (2005), Ban and Kalies (2006), Giesl (2007), Lazar and Jokić (2010), Lazar et al. (2013), Giesl and Hafstein (2014a) and the review in Giesl and Hafstein (2015).

Based on the method presented in Hafstein (2018), which was an extension of the method in Giesl and Hafstein (2014a) to discrete-time, switched systems with arbitrary switching, the authors developed in Palacios Roman and Hafstein (2024) a method to compute convex Lyapunov functions for discrete-time, switched linear systems using linear programming (LP). It was shown that given such a system, an LP problem could be constructed such that any feasible solution to the LP problem parameterizes a cone-wise linear Lyapunov function for the system, which asserts the exponential stability of the origin. In this paper we prove that, if the origin is an exponentially stable equilibrium for a two-dimensional switched linear system with arbitrary switching, then the LP problem possesses a feasible solution. Our proof only works for two-dimensional systems and we discuss in detail the challenges in higher dimensions.

The paper is structured as follows: First, in Section 2, we give a short introduction on discrete-time, switched linear systems, global exponential stability of the origin, and its characterization using Lyapunov functions. Second, in Section 3, we give a short description of the algorithm in Palacios Roman and Hafstein (2024) to compute convex Lyapunov functions for switched, discrete-time, linear systems. Third, we prove in Section 4 that, if the origin is globally exponentially stable for the switched system, then the LP problem generated by the algorithm in Section 3 is feasible, provided that the system dimension is $n = 2$ and the triangulation, used to parameterize the Lyapunov function, is sufficiently dense. Further, we discuss why the proof does not work in higher dimensions. Last, we provide some concluding remarks in Section 5.

2. SWITCHED LINEAR SYSTEMS, EXPONENTIAL STABILITY, AND LYAPUNOV FUNCTIONS

Consider the class of discrete-time, switched linear systems with arbitrary switching

$$\mathbf{x}_{k+1} \in \{A_\ell \mathbf{x}_k \colon A_\ell \in \mathbb{R}^{n\times n},\ \ell \in \mathcal{M}\}, \tag{1}$$

where $\mathcal{M} := \{1, 2, \ldots, M\}$ and $M \in \mathbb{N}_+ := \{1, 2, \ldots\}$. A solution trajectory to (1) starting at $\mathbf{x} \in \mathbb{R}^n$ at time $k = 0$ is denoted by $k \mapsto \boldsymbol{\psi}(k, \mathbf{x})$, i.e., $\boldsymbol{\psi}(0, \mathbf{x}) = \mathbf{x}$ and $\boldsymbol{\psi}(k + 1, \mathbf{x}) = A_{s(k)}\boldsymbol{\psi}(k, \mathbf{x})$, where $s\colon \mathbb{N}_0 \mapsto \mathcal{M}$ is an arbitrary switching signal and $k \in \mathbb{N}_0 := \{0, 1, \ldots\}$.

To study the strong stability of (1), i.e., stability for all switching signals, we use the following definitions of global exponential stability (GES) and Lyapunov functions.

Definition 1. The origin is said to be GES for the system (1), if there exist constants $C \geq 1$, $\alpha > 0$, such that for every initial condition $\mathbf{x} \in \mathbb{R}^n$ all solutions $\boldsymbol{\psi}(k, \mathbf{x})$ satisfy

$$\|\boldsymbol{\psi}(k, \mathbf{x})\| \leq Ce^{-\alpha k}\|\mathbf{x}\| \quad \text{for all} \ \ k \in \mathbb{N}_0, \tag{2}$$

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where $\|\cdot\|$ can be an arbitrary norm on $\mathbb{R}^n$. □

The GES of the origin for the system (1) is characterized by the existence of a particular Lyapunov function for the system, as defined in the next definition.

Definition 2. A continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a Lyapunov function for the system (1), if there exist constants $a, b, c > 0$ such that for all $\mathbf{x} \in \mathbb{R}^n$ we have

$$a\|\mathbf{x}\| \leq V(\mathbf{x}) \leq b\|\mathbf{x}\| \tag{3}$$

and

$$\max_{\ell \in \mathcal{M}} V(A_\ell \mathbf{x}) - V(\mathbf{x}) \leq -c\|\mathbf{x}\|. \tag{4}$$

□

It is easy to see that the existence of a Lyapunov function as in Definition 2 for the system (1) implies that the origin is GES for the system (1). Simply note that

$$V(\boldsymbol{\psi}(1, \mathbf{x})) - V(\mathbf{x}) \leq -c\|\mathbf{x}\| \quad \text{and} \quad V(\mathbf{x}) \leq b\|\mathbf{x}\|$$

imply that (note that necessarily $b \geq c$)

$$V(\boldsymbol{\psi}(1, \mathbf{x})) \leq \left(1 - \frac{c}{b}\right) V(\mathbf{x}),$$

from which it follows that

$$V(\boldsymbol{\psi}(k, \mathbf{x})) \leq \left(1 - \frac{c}{b}\right)^k V(\mathbf{x})$$

and

$$\|\boldsymbol{\psi}(k, \mathbf{x})\| \leq \frac{b}{a}\left(1 - \frac{c}{b}\right)^k \|\mathbf{x}\|,$$

which is equivalent to (2) with $C = b/a$ and $e^{-\alpha} = 1 - c/b$.

The existence of a (strictly) convex Lyapunov function as in Definition 2 for any GES difference inclusion is also well known, see Molchanov and Pyatnitskiy (1989). For completeness, a sketch of the construction of such a Lyapunov function is included below.

For every $\mathbf{i} = (i_1, i_2, \ldots, i_k) \in \mathcal{M}^k$, $k \in \mathbb{N}_+$, we have the solution trajectory

$$\boldsymbol{\psi}(k, \mathbf{x}) = A_{\mathbf{i}}\mathbf{x}, \quad \text{where} \quad A_{\mathbf{i}} := A_{i_k} A_{i_{k-1}} \cdots A_{i_1},$$

from the intitial position $\mathbf{x} \in \mathbb{R}^n$. The bound (2) implies that for every $\mathbf{x} \neq \mathbf{0}$ and every $\mathbf{i} \in \mathcal{M}^k$

$$\frac{\|A_{\mathbf{i}}\mathbf{x}\|}{\|\mathbf{x}\|} \leq Ce^{-\alpha k}.$$

In particular, $\|A_{\mathbf{i}}\| \leq C$, for all $\mathbf{i} \in \mathcal{M}^k$ and $k \in \mathbb{N}_+$, and there exists a $T \in \mathbb{N}_+$, $T \geq 2$, such that

$$\rho := \max_{\mathbf{i} \in \mathcal{M}^T} \|A_{\mathbf{i}}\| < 1.$$

Note that for $A \in \mathbb{R}^{n \times n}$, $\|A\|$ denotes the matrix norm induced by the vector norm $\|\cdot\|$, i.e., $\|A\| := \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$.

For every $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{i} \in \mathcal{M}^{T-1}$, we define

$$W_{\mathbf{i}}(\mathbf{x}) := \|\mathbf{x}\| + \|A_{i_1}\mathbf{x}\| + \|A_{i_2}A_{i_1}\mathbf{x}\| + \ldots + \|A_{\mathbf{i}}\mathbf{x}\|$$

and, our Lyapunov function,

$$W(\mathbf{x}) := \max_{\mathbf{i} \in \mathcal{M}^{T-1}} W_{\mathbf{i}}(\mathbf{x}). \tag{5}$$

Clearly, W is homogenous of degree one and the conditions (3) follow immediately with $a = 1$ and $b = TC$. To see property (4), note that for all $\mathbf{x} \in \mathbb{R}^n$ and $\ell \in \mathcal{M}$, where $\mathbf{i} \in \mathcal{M}^{T-1}$ such that $W(A_\ell \mathbf{x}) = W_{\mathbf{i}}(A_\ell \mathbf{x})$ and $\mathbf{j} = (\ell, i_1, i_2, \ldots, i_{T-1}) \in \mathcal{M}^T$, we have

$$\begin{aligned} W(A_\ell \mathbf{x}) &= \|A_\ell \mathbf{x}\| + \|A_{i_1}A_\ell \mathbf{x}\| + \ldots + \|A_{\mathbf{i}}A_\ell \mathbf{x}\| \\ &= W_{\mathbf{j}}(\mathbf{x}) - \|\mathbf{x}\| + \|A_{\mathbf{i}}A_\ell \mathbf{x}\| \\ &\leq W(\mathbf{x}) - \|\mathbf{x}\| + \rho\|\mathbf{x}\|, \end{aligned}$$

which implies (4) with $c = 1 - \rho > 0$.

The Lipschitz continuity of W follows from the reverse triangle inequality for norms:

$$\begin{aligned} W(\mathbf{x}) - W(\mathbf{y}) &\leq W_{\mathbf{i}}(\mathbf{x}) - W_{\mathbf{i}}(\mathbf{y}) \leq TC\|\mathbf{x} - \mathbf{y}\|, \\ W(\mathbf{y}) - W(\mathbf{x}) &\leq W_{\mathbf{j}}(\mathbf{y}) - W_{\mathbf{j}}(\mathbf{x}) \leq TC\|\mathbf{y} - \mathbf{x}\|, \end{aligned}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{i}, \mathbf{j} \in \mathcal{M}^{T-1}$ are such that $W(\mathbf{x}) = W_{\mathbf{i}}(\mathbf{x})$ and $W(\mathbf{y}) = W_{\mathbf{j}}(\mathbf{y})$.

Next, we show (strict) convexity: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\lambda \in (0, 1)$, and $\mathbf{i} \in \mathcal{M}^{T-1}$. Then, the triangle inequality implies

$$\begin{aligned} W_{\mathbf{i}}(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) &\leq \lambda W_{\mathbf{i}}(\mathbf{x}) + (1 - \lambda)W_{\mathbf{i}}(\mathbf{y}) \\ &\leq \lambda W(\mathbf{x}) + (1 - \lambda)W(\mathbf{y}), \end{aligned}$$

where the first $\leq$ can be replaced by $<$ if the norm $\|\cdot\|$ is strictly convex. Hence, W is (strictly) convex.

Note that since W is convex and homogenous of degree one, we have

$$W\left(\sum_{i=1}^{N} \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^{N} \lambda_i W(\mathbf{x}_i)$$

if $\lambda_i \geq 0$, $i \in \{1, 2, \ldots, N\}$ and $N \in \mathbb{N}_+$. Finally, since $W(\mathbf{x}) = W(-\mathbf{x})$, W is a norm on $\mathbb{R}^n$.

3. THE ALGORITHM

The algorithm in Palacios Roman and Hafstein (2024) to parameterize convex, cone-wise linear Lyapunov functions for (1), first constructs the Lyapunov function on a compact triangulated neighbourhood of the origin. This Lyapunov function is then extrapolated to the whole state-space $\mathbb{R}^n$. We give a short description of the triangulation and the parameterization of the Lyapunov function, before we state the LP problem that can parameterize such a Lyapunov function. For examples of Lyapunov functions computed with the algorithm see Palacios Roman and Hafstein (2024).

3.1 Triangulation

A triangulation $\mathcal{T}$ of a compact neighbourhood $\mathcal{D} \subset \mathbb{R}^n$ of the origin is a partition of $\mathcal{D}$ into n-simplices. An n-simplex $\mathfrak{S}_\nu \subset \mathbb{R}^n$ with vertices $\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu \in \mathbb{R}^n$, $\mathbf{x}_0^\nu = \mathbf{0}$, is defined as

$$\begin{aligned} \mathfrak{S}_\nu &= \operatorname{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu) \\ &:= \left\{ \sum_{i=0}^{n} \lambda_i \mathbf{x}_i^\nu : \sum_{i=0}^{n} \lambda_i = 1 \text{ and all } \lambda_i \geq 0 \right\}. \end{aligned}$$

We assume that the vectors $\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu$ are linearly independent and that the vertices of a simplex $\mathfrak{S}_\nu \in \mathcal{T}$ have a fixed order, which is why we write $\mathfrak{S}_\nu = \operatorname{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu)$ rather than $\operatorname{co}\{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\}$, i.e., ordered tuple rather than set. Such a simplex $\mathfrak{S}_\nu \in \mathcal{T}$ is always non-degenerate, i.e., it has a positive n-dimensional volume, and for every $\mathbf{x} \in \mathfrak{S}_\nu$ there is a unique set of numbers $0 \leq \lambda_i \leq 1$, $i \in \{0, 1, \ldots, n\}$, such that $\sum_{i=0}^n \lambda_i = 1$ and $\mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu$. The matrix

$$X_\nu := [\mathbf{x}_1^\nu \ \mathbf{x}_2^\nu \ \ldots \ \mathbf{x}_n^\nu] \in \mathbb{R}^{n \times n}$$

is invertible and well-defined for the simplex $\mathfrak{S}_\nu$.

Further, the triangulation $\mathcal{T}$ must be shape-regular, i.e., for any $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ in $\mathcal{T}$, $\nu \neq \mu$, it must hold that

$$\mathfrak{S}_\nu \cap \mathfrak{S}_\mu = \operatorname{co}(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_k), \quad \mathbf{y}_j = \mathbf{x}_{\ell_j^\nu}^\nu = \mathbf{x}_{\ell_j^\mu}^\mu,$$

where $\mathbf{y}_0 := \mathbf{0}$, $j \in \{0, 1, \ldots, k\}$, $0 \le k < n$, $\ell_j^\nu, \ell_j^\mu \in \{0, 1, \ldots, n\}$, and neither $\ell_j^\nu = \ell_m^\nu$ nor $\ell_j^\mu = \ell_m^\mu$ if $j \neq m$. In other words, $\mathcal{T}$ is shape-regular if any two n-simplices $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ in $\mathcal{T}$, $\nu \neq \mu$, intersect in a common lower dimensional face.

For every simplex $\mathfrak{S}_\nu \in \mathcal{T}$ a corresponding simplicial cone can be defined by

$$\mathfrak{C}_\nu := \left\{ \sum_{i=1}^n \lambda_i \mathbf{x}_i^\nu \colon \lambda_i \ge 0 \right\},$$

where we used that $\mathbf{x}_0^\nu = \mathbf{0}$. The set-theoretic union of the $\mathfrak{C}_\nu$ is equal to $\mathbb{R}^n$.

3.2 Parameterization of cone-wise linear Lyapunov functions

Given a triangulation $\mathcal{T}$ as defined above, a function $V \colon \mathbb{R}^n \to \mathbb{R}$ can be parameterized by specifying its values at all the vertices $\mathbf{x}$ in $\mathcal{T}$, where we refer to $\mathbf{x}$ as a vertex of $\mathcal{T}$ if $\mathbf{x}$ is a vertex of any simplex in $\mathcal{T}$. Let these values be denoted by $V_\mathbf{x}$, i.e., $V(\mathbf{x}) := V_\mathbf{x}$. Note that if two simplices $\mathfrak{S}_\nu, \mathfrak{S}_\mu \in \mathcal{T}$ have a common vertex $\mathbf{x}$, i.e., if $\mathbf{x} = \mathbf{x}_i^\nu = \mathbf{x}_j^\mu$ for some $i, j \in \{0, 1, \ldots, n\}$, then $V_{\mathbf{x}_i^\nu} = V_{\mathbf{x}_j^\mu}$. This ensures V is well-defined and continuous.

Note that for every $\mathbf{x} \in \mathbb{R}^n$ there exists a simplicial cone $\mathfrak{C}_\nu$ such that $\mathbf{x} \in \mathfrak{C}_\nu$, i.e., there exist $\lambda_i \ge 0$ such that $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{x}_i^\nu$. Therefore, the function V can be defined by

$$V(\mathbf{x}) := \sum_{i=1}^n \lambda_i V_{\mathbf{x}_i^\nu}. \tag{6}$$

This can equivalently be expressed as

$$V(\mathbf{x}) = \nabla V_\nu \bullet \mathbf{x} \qquad \text{if } \mathbf{x} \in \mathfrak{C}_\nu,$$

where

$$\nabla V_\nu = \mathbf{v}_\nu^\top X_\nu^{-1} \tag{7}$$

and

$$\mathbf{v}_\nu := \begin{bmatrix} V_{\mathbf{x}_1^\nu} & V_{\mathbf{x}_2^\nu} & \ldots & V_{\mathbf{x}_n^\nu} \end{bmatrix}^\top \in \mathbb{R}^n. \tag{8}$$

3.3 LP approach for constructing cone-wise linear Lyapunov functions

To find suitable values for the $V_\mathbf{x}$ such that (6) is a Lyapunov function for (1), an LP problem can be used. In Palacios Roman and Hafstein (2024) the following theorem is proved:

Theorem 1. Consider system (1) and a triangulation $\mathcal{T}$ as above. Let $c_1, c_2 > 0$, let $V_\mathbf{x} \in \mathbb{R}$ for every vertex $\mathbf{x}$ in $\mathcal{T}$ and let $V \colon \mathbb{R}^n \to \mathbb{R}$ be parameterized by the values $V_\mathbf{x}$ as defined above. Assume the following conditions are fulfilled:

(i) V is zero at the origin, i.e.,

$$V_\mathbf{0} = 0. \tag{9}$$

(ii) For every vertex $\mathbf{x}$ in $\mathcal{T}$

$$V_\mathbf{x} \ge c_1 \|\mathbf{x}\|. \tag{10}$$

(iii) For every simplex $\mathfrak{S}_\nu$ in $\mathcal{T}$ and for every nonzero vertex $\mathbf{x}_i^\nu$ of $\mathfrak{S}_\nu$, i.e., for all $i \in \{1, 2, \ldots, n\}$, we have

$$V(A_\ell \mathbf{x}_i^\nu) - V_{\mathbf{x}_i^\nu} \le -c_2 \|\mathbf{x}_i^\nu\| \tag{11}$$

for every $\ell \in \mathcal{M}$.

(iv) For every two simplices $\mathfrak{S}_\nu$, $\mathfrak{S}_\mu \in \mathcal{T}$ with $\mathfrak{S}_\nu \cap \mathfrak{S}_\mu \neq \{\mathbf{0}\}$, i.e., $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ have at least two common vertices, and for all $i \in \{1, 2, \ldots, n\}$

$$[\nabla V_\nu - \nabla V_\mu] \bullet \mathbf{x}_i^\nu \ge 0 \text{ and } [\nabla V_\mu - \nabla V_\nu] \bullet \mathbf{x}_i^\mu \ge 0. \tag{12}$$

Then V is a convex, cone-wise linear Lyapunov function for system (1).

□

Remark 1. Note that the inequalities in (12) are trivially fulfilled for the common vertices of $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$. Hence, those constraints need not be checked in the LP problem.

The values $V_\mathbf{x}$ are the variables of the LP problem and (9), (10), (11), and (12) are its constraints. The theorem states that any feasible solution to the LP problem parameterizes a Lyapunov function for the system (1).

4. MAIN RESULT

For our proof, we require a suitable concrete triangulation for the LP problem to parameterize our Lyapunov function. We will use the triangular-fan $\mathcal{T}_K$ of the triangulation in Giesl and Hafstein (2014b) and Andersen et al. (2023). The density of the triangulation is determined through $K \in \mathbb{N}_+$ and we show that if the origin is GES for (1), then for every K sufficiently large, the LP problem in Theorem 1 has a feasible solution. In more detail, we show that if we set $V_\mathbf{x} = sW(\mathbf{x})$, $s > 0$, where W is the Lyapunov function in (5), then the constraints are all fulfilled: The constraints (9) and (10) are always fulfilled, for any n and K. The constraints (11) are fulfilled for any n if K is sufficiently large. For the constraints (12) to be fulfilled, we do not need K to be sufficiently large, but we need to assume in our proof that $n = 2$.

4.1 The triangulation $\mathcal{T}_K$

In the definition of the triangulation $\mathcal{T}_K$ we use the functions $\mathbf{R}^\mathcal{J} : \mathbb{R}^n \to \mathbb{R}^n$, defined for every $\mathcal{J} \subset \{1, 2, \ldots, n\}$ by

$$\mathbf{R}^\mathcal{J}(\mathbf{x}) := \sum_{i=1}^n (-1)^{\chi_\mathcal{J}(i)} x_i \mathbf{e}_i, \quad \chi_\mathcal{J}(i) := \begin{cases} 1, & \text{if } i \in \mathcal{J}, \\ 0, & \text{if } i \notin \mathcal{J}. \end{cases}$$

where $\mathbf{e}_i$ is the standard ith unit vector in $\mathbb{R}^n$. Thus, $\mathbf{R}^\mathcal{J}(\mathbf{x})$ is the vector $\mathbf{x}$, except for a minus has been put in front of the coordinate x_i whenever $i \in \mathcal{J}$.

Firstly, we define the standard triangulation $\mathcal{T}^{\text{std}}$, which consists of the simplices

$$\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma} := \text{co}\left(\mathbf{x}_0^{\mathbf{z}\mathcal{J}\sigma}, \mathbf{x}_1^{\mathbf{z}\mathcal{J}\sigma}, \ldots, \mathbf{x}_n^{\mathbf{z}\mathcal{J}\sigma}\right),$$

where

$$\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma} := \mathbf{R}^\mathcal{J}\left(\mathbf{z} + \sum_{i=1}^j \mathbf{e}_{\sigma(i)}\right)$$

for all $\mathbf{z} \in \mathbb{N}_0^n$, all $\mathcal{J} \subset \{1, 2, \ldots, n\}$, all permutations σ of $\{1, 2, \ldots, n\}$, and all $j \in \{0, 1, \ldots, n\}$. Note that $\mathbf{e}_{\sigma(i)}$, $j = \sigma(i)$, is the standard jth unit vector.

Now fix a $K \in \mathbb{N}_+$ and consider the simplices $\mathfrak{S}_{\mathbf{z}\mathcal{J}\sigma}$ in $\mathcal{T}^{\text{std}}$ that are a subset of $[-K, K]^n \subset \mathbb{R}^n$ and intersect the boundary of the hypercube $[-K, K]^n$. We are only interested in those intersections that are $(n-1)$-simplices. Thus, we take every simplex with vertices $\mathbf{x}_j^{\mathbf{z}\mathcal{J}\sigma}$, $j \in \{0, 1, \ldots, n\}$,

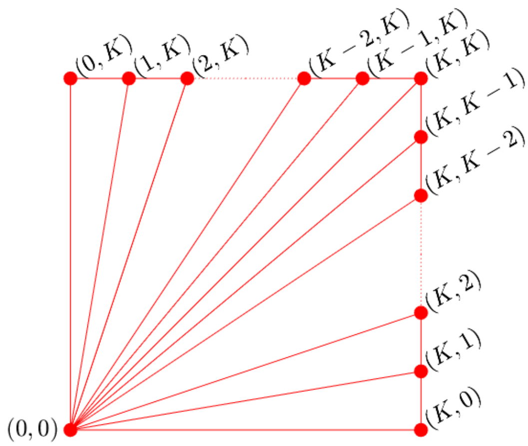

Fig. 1. Visualization of the triangulation $\mathcal{T}_K$ of $\mathbb{R}^2$ in the first quadrant. The vertices of $\mathcal{T}_K$ are indicated and each triangle represents a simplex of $\mathcal{T}_K$

where exactly one vertex $\mathbf{x}_{j^*}^{\mathbf{z}\mathcal{J}\sigma}$ satisfies $\|\mathbf{x}_{j^*}^{\mathbf{z}\mathcal{J}\sigma}\|_\infty < K$ and $\|\mathbf{x}_{j}^{\mathbf{z}\mathcal{J}\sigma}\|_\infty = K$ for $j \in \{0, 1, \dots, n\} \setminus \{j^*\}$. Then we replace the vertex $\mathbf{x}_{j^*}^{\mathbf{z}\mathcal{J}\sigma}$ by $\mathbf{0}$. Note that j^* is necessarily equal to 0. The set of the simplices constructed in this way triangulates $[-K, K]^n$ and is denoted by $\mathcal{T}_K$. A visualization of the triangulation is given in Fig. 1.

4.2 Proof of convergence

It is more convenient in our proof to scale $\mathcal{T}_K$ down to the hypercube $[-1, 1]^n$. Corresponding to the simplex $\mathfrak{S}_\nu = \operatorname{co}(\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \dots, \mathbf{x}_n^\nu) \in \mathcal{T}_K$, we define the simplex $\operatorname{co}(K^{-1}\mathbf{x}_0^\nu, K^{-1}\mathbf{x}_1^\nu, \dots, K^{-1}\mathbf{x}_n^\nu) \in K^{-1}\mathcal{T}_K$ and the triangulation $K^{-1}\mathcal{T}_K$ consist of exactly the simplices obtained in this manner from the simplices in $\mathcal{T}_K$. Note that the vertices $\mathbf{x}_k^\nu$ of $K^{-1}\mathcal{T}_K$ fulfill

$$\|\mathbf{x}_k^\nu\|_\infty = 1 \quad \text{and} \quad \|\mathbf{x}_k^\nu - \mathbf{x}_\ell^\nu\|_\infty \leq \frac{1}{K}$$

for all $\mathfrak{S}_\nu \in K^{-1}\mathcal{T}_K$ and $k, \ell \in \{1, 2, \dots, n\}$.

Theorem 2. Assume that the origin is GES for system (1) and $n = 2$. Then there exists a $K^* \in \mathbb{N}_+$, such that for every $K \in \mathbb{N}_+$, $K \geq K^*$, the LP problem in Theorem 1 has a feasible solution when using the triangulation $K^{-1}\mathcal{T}_K$.

Proof Let W be the Lyapunov function (5) for the system (1) and recall that $W(\mathbf{x}) \geq \|\mathbf{x}\|$ and $W(A_\ell\mathbf{x}) - W(\mathbf{x}) \leq -(1-\rho)\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$ and $\ell \in \mathcal{M}$. Let $L > 0$ be a Lipschitz constant for W with respect to the $\|\cdot\|_\infty$ norm, i.e., $|W(\mathbf{x}) - W(\mathbf{y})| \leq L\|\mathbf{x} - \mathbf{y}\|_\infty$, and let $\beta > 0$ be a constant such that $\|\mathbf{x}\|_\infty \leq \beta\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.

Set

$$s := \max\left\{c_1, \frac{2c_2}{1-\rho}\right\},$$

where $c_1, c_2 > 0$ are the constants from the LP problem in Theorem 1 and fix $K^* \in \mathbb{N}_+$ such that

$$K^* \geq \frac{s\beta L}{c_2}. \tag{13}$$

For every vertex $\mathbf{x}$ of the triangulation $K^{-1}\mathcal{T}_K$ we set $V_\mathbf{x} = sW(\mathbf{x})$ and show that all the constraints of the LP problem in Theorem 1 are fulfilled if $K \geq K^*$.

The constraints (9) and (10) are automatically fulfilled, because $V_\mathbf{0} = sW(\mathbf{0}) = 0$ and $V_\mathbf{x} = sW(\mathbf{x}) \geq c_1\|\mathbf{x}\|$ for all vertices $\mathbf{x} \in K^{-1}\mathcal{T}_K$. Note that this is independent of both K and n.

We now prove that the constraints (11) are fulfilled. To this end, first note that for an arbitrary $\mathbf{y} \in \mathfrak{C}_\nu$ we have $\mathbf{y} = \sum_{i=1}^n \lambda_i\mathbf{x}_i^\nu$, $\lambda_i \geq 0$, where $\mathbf{x}_i^\nu$ are the vertices of $\mathfrak{S}_\nu$. Because $\|\mathbf{x}_i^\nu\|_\infty = 1$ for $i \in \{1, 2, \dots, n\}$ and there exists a coordinate $j^* \in \{1, 2, \dots, n\}$ such that either $[\mathbf{x}_i^\nu]_{j^*} = 1$ or $[\mathbf{x}_i^\nu]_{j^*} = -1$ for all $i \in \{1, 2, \dots, n\}$, we have that

$$\begin{aligned}
\|\mathbf{y}\|_\infty &= \left\|\sum_{i=1}^n \lambda_i\mathbf{x}_i^\nu\right\|_\infty = \max_{j \in \{1,2,\dots,n\}} \left|\sum_{i=1}^n \lambda_i\mathbf{e}_j^\top\mathbf{x}_i^\nu\right| \\
&= \left|\sum_{i=1}^n \lambda_i\mathbf{e}_{j^*}^\top\mathbf{x}_i^\nu\right| = \left|\sum_{i=1}^n \lambda_i[\mathbf{x}_i^\nu]_{j^*}\right| = \\
&\left|\sum_{i=1}^n \lambda_i\right| = \sum_{i=1}^n \lambda_i \quad \vee \quad \left|\sum_{i=1}^n -\lambda_i\right| = \sum_{i=1}^n \lambda_i.
\end{aligned}$$

We define, for all $\mathbf{y} \neq \mathbf{0}$, $S := \sum_{i=1}^n \lambda_i > 0$ and $\widetilde{\lambda}_i = \lambda_i/S$ for $i \in \{1, 2, \dots, n\}$. Then, $\sum_{i=1}^n \widetilde{\lambda}_i = 1$ and by using the homogeneity of W, we get

$$\begin{aligned}
&\sum_{i=1}^n \lambda_i W(\mathbf{x}_i^\nu) - W\left(\sum_{i=1}^n \lambda_i\mathbf{x}_i^\nu\right) \\
&\quad = S\left[\sum_{i=1}^n \widetilde{\lambda}_i W(\mathbf{x}_i^\nu) - W\left(\sum_{i=1}^n \widetilde{\lambda}_i\mathbf{x}_i^\nu\right)\right] \\
&\quad = S\left[\sum_{i=1}^n \widetilde{\lambda}_i W(\mathbf{x}_i^\nu) - \sum_{i=1}^n \widetilde{\lambda}_i W\left(\sum_{j=1}^n \widetilde{\lambda}_j\mathbf{x}_j^\nu\right)\right] \\
&\quad \leq S\sum_{i=1}^n \widetilde{\lambda}_i \left|W(\mathbf{x}_i^\nu) - W\left(\sum_{j=1}^n \widetilde{\lambda}_j\mathbf{x}_j^\nu\right)\right| \\
&\quad \leq S\sum_{i=1}^n \widetilde{\lambda}_i L \left\|\mathbf{x}_i^\nu - \sum_{j=1}^n \widetilde{\lambda}_j\mathbf{x}_j^\nu\right\|_\infty \\
&\quad = SL\sum_{i=1}^n \widetilde{\lambda}_i \left\|\sum_{j=1}^n \widetilde{\lambda}_j\mathbf{x}_i^\nu - \sum_{j=1}^n \widetilde{\lambda}_j\mathbf{x}_j^\nu\right\|_\infty \\
&\quad \leq SL\sum_{i=1}^n \widetilde{\lambda}_i \sum_{j=1}^n \widetilde{\lambda}_j \|\mathbf{x}_i^\nu - \mathbf{x}_j^\nu\|_\infty \\
&\quad \leq \|\mathbf{y}\|_\infty \cdot L \cdot \frac{1}{K} \\
&\quad \leq \frac{\beta L}{K}\|\mathbf{y}\|.
\end{aligned}$$

Now, with $\mathbf{x}$ a nonzero vertex of $K^{-1}\mathcal{T}_K$ and $\mathbf{y} = A_\ell\mathbf{x} \in \mathfrak{C}_\nu$, $\ell \in \mathcal{M}$, we get

$$V(A_\ell\mathbf{x}) = V\left(\sum_{i=1}^n \lambda_i\mathbf{x}_i^\nu\right) = \sum_{i=1}^n \lambda_i V_{\mathbf{x}_i^\nu} = s\sum_{i=1}^n \lambda_i W(\mathbf{x}_i^\nu)$$

and

$$
\begin{aligned}
&V(A_\ell\mathbf{x}) - V(\mathbf{x})\\
&= V(A_\ell\mathbf{x}) - sW(A_\ell\mathbf{x}) + sW(A_\ell\mathbf{x}) - V(\mathbf{x})\\
&= s\left[\sum_{i=1}^n \lambda_i W(\mathbf{x}_i^\nu) - W\left(\sum_{i=1}^n \lambda_i \mathbf{x}_i^\nu\right) + W(A_\ell\mathbf{x}) - W(\mathbf{x})\right]\\
&\le s\left[\frac{\beta L}{K}\|\mathbf{x}\| - (1-\rho)\|\mathbf{x}\|\right] \le -c_2\|\mathbf{x}\|,
\end{aligned}
$$

because $-s(1-\rho) \le -2c_2$ and $s\beta L/K \le c_2$ if $K \ge K^*$. Thus, the constraints (11) are fulfilled when $K \ge K^*$.

To prove that the constraints (12) are fulfilled we assume $n = 2$. Let $\mathfrak{S}_\nu,\ \mathfrak{S}_\mu \in \mathcal{T}$ with $\mathfrak{S}_\nu \cap \mathfrak{S}_\mu \neq \{\mathbf{0}\}$, and let $\mathbf{x}_i^\nu \notin \mathfrak{S}_\mu$. We must show that

$$[\nabla V_\nu - \nabla V_\mu] \bullet \mathbf{x}_i^\nu \ge 0.$$

Now, by (7) and (8), we have

$$\nabla V_\nu = \mathbf{v}_\nu^\top X_\nu^{-1} \quad \text{and} \quad \nabla V_\mu = \mathbf{v}_\mu^\top X_\mu^{-1},$$

so

$$
\begin{aligned}
[\nabla V_\nu - \nabla V_\mu] \bullet \mathbf{x}_i^\nu &= \mathbf{v}_\nu^\top \mathbf{e}_i - \mathbf{v}_\mu^\top X_\mu^{-1}\mathbf{x}_i^\nu \qquad (14)\\
&= V_{\mathbf{x}_i^\nu} - r_1 V_{\mathbf{x}_1^\mu} - r_2 V_{\mathbf{x}_2^\mu}\\
&= s\cdot\left[W(\mathbf{x}_i^\nu) - r_1 W(\mathbf{x}_1^\mu) - r_2 W(\mathbf{x}_2^\mu)\right],
\end{aligned}
$$

where $\mathbf{r} = (r_1, r_2)^\top$ is the solution to

$$r_1\mathbf{x}_1^\mu + r_2\mathbf{x}_2^\mu = X_\mu\mathbf{r} = \mathbf{x}_i^\nu.$$

Note that at least one of r_1 or r_2 must be negative, because otherwise $\mathbf{x}_i^\nu \in \mathfrak{C}_\mu \cap [-1,1]^n = \mathfrak{S}_\mu$ which would contradict $\mathbf{x}_i^\nu \notin \mathfrak{S}_\mu$. Let us assume w.l.o.g. that $r_1 < 0$. Then,

$$\mathbf{x}_i^\nu + (-r_1)\mathbf{x}_1^\mu = r_2\mathbf{x}_1^\mu$$

and by the convexity and homogeneity of W it follows that

$$
\begin{aligned}
W(\mathbf{x}_i^\nu) + (-r_1)W(\mathbf{x}_1^\mu) &\ge W\left(\mathbf{x}_i^\nu + (-r_1)\mathbf{x}_1^\mu\right)\\
&= W(r_2\mathbf{x}_1^\mu)\\
&= r_2 W(\mathbf{x}_1^\mu),
\end{aligned}
$$

i.e.,

$$W(\mathbf{x}_i^\nu) - r_1 W(\mathbf{x}_1^\mu) - r_2 W(\mathbf{x}_2^\mu) \ge 0.$$

Thus, the right-hand-side of (14) is nonnegative and the constraints (11) are fulfilled.

□

Remark 2. For this line of reasoning to work, we need the solution $\mathbf{r} \in \mathbb{R}^n$ to $X_\mu\mathbf{r} = \mathbf{x}_i^\nu$ to have at most one component larger than zero. Unfortunately, this is not true for the triangulation $K^{-1}\mathcal{T}_K$ for $n > 2$. As a counterexample, consider the two simplices

$$\mathfrak{S}_1 = \operatorname{co}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}K^{-1}\\0\\1\end{pmatrix}, \begin{pmatrix}K^{-1}\\K^{-1}\\1\end{pmatrix}\right\}$$

and

$$\mathfrak{S}_2 = \operatorname{co}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\K^{-1}\\1\end{pmatrix}, \begin{pmatrix}-K^{-1}\\K^{-1}\\1\end{pmatrix}\right\},$$

which both are in $K^{-1}\mathcal{T}_K$ and have two vertices in common. Since

$$\begin{pmatrix}0\\K^{-1}\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}K^{-1}\\0\\1\end{pmatrix} + \begin{pmatrix}K^{-1}\\K^{-1}\\1\end{pmatrix},$$

the solution to $X_1\mathbf{r} = \mathbf{x}_2^2$ is $\mathbf{r} = [1\ -1\quad 1]^\top$, which has more than one positive component.

□

Remark 3. To prove convergence for $n > 2$, one might try to express (14) as

$$
\begin{aligned}
[\nabla V_\nu - \nabla V_\mu] \bullet \mathbf{x}_i^\nu &= \mathbf{v}_\nu^\top \mathbf{e}_i - \mathbf{v}_\mu^\top X_\mu^{-1}\mathbf{x}_i^\nu\\
&= s\cdot\left[W(\mathbf{x}_i^\nu) - \sum_{j=1}^n r_j W(\mathbf{x}_j^\mu)\right],
\end{aligned}
$$

where $X_\mu\mathbf{r} = \mathbf{x}_i^\nu$. Then, by convexity of W,

$$
\begin{aligned}
W(\mathbf{x}_i^\nu) - \sum_{\substack{j=1\\ r_j<0}}^n r_j W(\mathbf{x}_j^\mu) &= W\left(\mathbf{x}_i^\nu - \sum_{\substack{j=1\\ r_j<0}}^n r_j \mathbf{x}_j^\mu\right) + \varepsilon_1\\
&= W\left(\sum_{\substack{j=1\\ r_j\ge 0}}^n r_j \mathbf{x}_j^\mu\right) + \varepsilon_1\\
&= \sum_{\substack{j=1\\ r_j\ge 0}}^n r_j W(\mathbf{x}_j^\mu) - \varepsilon_2 + \varepsilon_1,
\end{aligned}
$$

where $\varepsilon_1, \varepsilon_2 \ge 0$. If we can show that $\varepsilon_1 \ge \varepsilon_2$ if K is sufficiently large, the proof of convergence for $n > 2$ would be complete. For this, one might want to choose the norm in (5) cleverly or use a different Lyapunov function W altogether, e.g., a Lyapunov functions constructed using the method in Yoshizawa (1966), as in Geiselhart (2016), rather than the method in Kurzweil (1963) or Massera (1956), as we did.

Another attempt to tackle the case $n > 2$ might be to consider if one really needs the constraints (12) for all pairs of simplices with $\mathfrak{S}_\nu \cap \mathfrak{S}_\mu \neq \{\mathbf{0}\}$ for the triangulation $K^{-1}\mathcal{T}_K$. From the proof of Lemma 1 in Palacios Roman and Hafstein (2024), which asserts that the parameterized function V is convex, it is eminent that for a very regular triangulation like $K^{-1}\mathcal{T}_K$ the constraints (12) are not necessary for some such pairs of simplices for V to be convex.

Remark 4. The sufficient lower bound K^* in (13) on K for the LP problem in Theorem 1 to be feasible should not be used in practice, as it will in general be much larger than necessary. A better approach is just to start with some particular $K \in \mathbb{N}_+$ and if there is not a feasible solution to the LP problem, then increase K and try again. Theorem 2 guaranties that if the origin is GES for the system, then this strategy will be successful. For this reason, we do not attempt to derive a formula in the original data, i.e. the matrices A_ℓ, in the bound (13). An explicit bound in the original data on triangulations for LP problems to compute Lyapunov functions for continuous-time nonlinear systems has been derived in Hafstein (2004), but is so conservative that it is useless in practice.

5. CONCLUSION

In this paper, we proved that the algorithm in Palacios Roman and Hafstein (2024) to parameterize a convex Lyapunov function for discrete-time, switched linear systems under arbitrary switching with globally exponentially stable equilibrium, always succeeds if the dimension of the system is $n = 2$ and the density of the triangulation is sufficiently high. Additionally, we discussed in some detail why the argumentation of the proof does not work in higher dimensions and possible directions that might lead to a proof for $n > 2$. Such a proof is a work in progress.

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