Smoothing homogenous Lyapunov functions ⋆

Sigurdur F. Hafstein

Faculty of Physical Sciences, University of Iceland, Reykjavik, Iceland (e-mail: shafstein@hi.is)

Abstract: Dayawansa and Martin proved in 1999 that locally Lipschitz continuous and homogenous Lyapunov functions for a switched linear systems can be smoothed to C^∞ Lyapunov functions retaining the homogeneity. Their proof used some rather advanced concepts in differential geometry. In this paper we give a more elementary proof and, additionally, show that our smooth Lyapunov function and its orbital derivatives approximate the original Lyapunov function and its orbital derivatives arbitrary close and that the smoothing technique preserves symmetry of the Lyapunov functions. These additional properties of the smooth Lyapunov function are useful, for example, when studying numerical methods to compute Lyapunov functions. Finally, our proof works for switched nonlinear systems, provided the individual subsystems have globally Lipschitz continuous right-hand sides.

Keywords: Homogenous Lyapunov functions, switched systems, differential inclusions, smoothing of functions.

1. INTRODUCTION

In Section III in the seminal paper Dayawansa and Martin (1999) it was proved, amongst other things, that a locally Lipschitz continuous, homogenous of order two Lyapunov function $V\colon \mathbb{R}^n \to \mathbb{R}$ for a switched linear system can be smoothed to a homogenous of order two Lyapunov function that is C^∞ except at the origin. However, the proof uses some quite advanced concepts of differential geometry, like integration over the special orthogonal matrices $SO(n)$ using the Haar measure Haar (1933) and some details are left out in the proof. Further, for some applications it would be advantageous to have stronger statements about the difference between the orbital derivatives of the original locally Lipschitz Lyapunov function and the smooth Lyapunov function constructed; such statements were not needed for the application in Dayawansa and Martin (1999).

In this paper, we deliver a more elementary proof and we prove that our smooth Lyapunov function and its orbital derivatives are arbitrary close to the original Lyapunov function and its orbital derivatives. Further, we show that our smoothing technique preserves symmetry of the Lyapunov function, i.e. if $V(x) = V(-x)$ for the original Lipschitz continuous Lyapunov function, then the same holds true for the smooth approximation. Finally, our proof works for switched nonlinear systems, provided the individual subsystems have globally Lipschitz continuous right-hand sides.

We use quite elementary differential geometry in our proof, all of which are covered in the classic book *Analysis on Manifolds* Munkres (1991), and we work out the proof in detail with citations to the relevant results in this book. Since we are interested in computational methods for Lyapunov functions, that parameterize piecewise linear Lyapunov functions using linear programming, see e.g. Polanski (1997); Della Rossa et al. (2020); Andersen et al. (2023a,b); Hafstein (2023); Hafstein and Tanwani (2023), we concentrate on Lyapunov functions that are homogenous of order one, rather than of order two as in Dayawansa and Martin (1999).

In the next section we describe the problem setting in more detail and state the main results. In Section 3 we prove the main result and in Section 4 we conclude the paper.

2. MAIN RESULTS

Let $g_i\colon \mathbb{R}^n \to \mathbb{R}^n$, $g_i(0) = 0$, be globally Lipschitz continuous functions for $i = 1, 2, \dots, N$. That is, there exists a common Lipschitz constant $G > 0$ such that for all $x, y \in \mathbb{R}^n$ we have

$$\|g_i(x) - g_i(y)\| \le G\|x - y\|, \quad i = 1, 2, \dots, N, \tag{1}$$

where $\|\cdot\|$ denotes the Euclidian norm. We consider the arbitrary switched system

$$\dot{x} = g_\sigma(x), \tag{2}$$

see, e.g., Davrazos and Koussoulas (2001); Liberzon (2003); Sun and Ge (2011). This means that $\sigma\colon [0,\infty) \to \{1, 2, \dots, N\}$, the *switching signal*, is an arbitrary right-continuous function with only finitely many discontinuity points on every compact interval. A solution $x(t)$ to (2) is obtained by gluing continuously together solution trajectory segments of

$$x_t^i := x + \int_0^t g_i(x_\tau^i)d\tau \quad \text{for } t \ge 0. \tag{3}$$

That is, with initial-value $\xi \in \mathbb{R}^n$ at time $t_0 = 0$, we set $x(t) = x_{t-t_0}^i$ in (3) with $x = \xi$ and $i = i_0 := \sigma(t_0)$ for $t_0 \le t \le t_1$, where $t_1 > 0$ is the first discontinuity point

⋆ This work was supported in part by the Icelandic Research Fund under Grant 228725-051

of σ. On the interval $[t_1, t_2]$, where $t_2 > t_1$ is the second discontinuity point of σ, we set $x(t) = x^i_{t-t_1}$ in (3) with $x = x^{i_0}_{t_1-t_0}$ and $i = i_1 := \sigma(t_1)$ for $t_1 \le t \le t_2$, etc.

We assume that the system (2) possesses a locally Lipschitz continuous Lyapunov function $V: \mathbb{R}^n \to \mathbb{R}$, that is homogenous of order one. In detail, that there exist constants $a, b, c > 0$, such that for all $x \in \mathbb{R}^n$ we have

$$a\|x\| \le V(x) \le b\|x\|, \quad V(sx) = sV(x) \text{ for all } s > 0,$$

and, with x^i_h defined in (3),

$$\limsup_{h\to 0+} \frac{V(x^i_h) - V(x)}{h} \le -c\|x\| \tag{4}$$

for $i = 1, 2, \ldots, N$ and $x \ne 0$.

Remark 1. Since V is locally Lipschitz and homogenous of order one, it is indeed globally Lipschitz. To see this let $L > 0$ be a Lipschitz constant for V on the closed unit ball around zero. With $x, y \in \mathbb{R}^n$, not both the zero vector, set $s := \max\{\|x\|, \|y\|\} > 0$ and note that

$$\begin{aligned} |V(x) - V(y)| &= s|V(x/s) - V(y/s)| \\ &\le sL\|x/s - y/s\| \\ &\le L\|x - y\| \end{aligned}$$

and L is a global Lipschitz constant for V.

We will show, that given these assumptions, we have:

Theorem 1. For every $\varepsilon > 0$ there exists a Lyapunov function $V_\varepsilon: \mathbb{R}^n \to \mathbb{R}$ that is C^∞ on $\mathbb{R}^n \setminus \{0\}$, homogenous of order one, and such that

$$|V_\varepsilon(x) - V(x)| \le \varepsilon\|x\| \tag{5}$$

for all $x \in \mathbb{R}^n$ and

$$\nabla V_\varepsilon(x) g_i(x) \le -(c - \varepsilon)\|x\|$$

for $i = 1, 2, \ldots, N$ and $x \ne 0$.

Further, if $V(x) = V(-x)$ for all $x \in \mathbb{R}^n$, then $V_\varepsilon(x) = V_\varepsilon(-x)$ for all $x \in \mathbb{R}^n$.

An obvious corollary is that

$$(a - \varepsilon)\|x\| \le V_\varepsilon(x) \le (b + \varepsilon)\|x\|$$

for all $x \in \mathbb{R}^n$ and that

$$\limsup_{h\to 0+} \frac{V_\varepsilon(x^i_h) - V_\varepsilon(x)}{h} \le -(c - \varepsilon)\|x\|$$

for $i = 1, 2, \ldots, N$ and $x \ne 0$.

3. PROOF OF THE MAIN RESULTS

The idea of the proof is as follows: First we smooth out V on the unit sphere $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ using a smooth mollifier. We do this using smooth, local, coordinate patches ϕ_α from $U_\alpha \subset S^{n-1}$ to $\mathbb{R}^{n-1}$, i.e. a smooth atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ for the manifold S^{n-1}. For this atlas there exists a corresponding partition of unity $\psi_\alpha: S^{n-1} \to [0, 1]$ with

$$\operatorname{supp}(\psi_\alpha) := \overline{\{x \in S^{n-1} : \psi_\alpha(x) \ne 0\}} \subset U_\alpha.$$

For each U_α we so obtain a smooth approximation V_α of V on an open set $U_\alpha \supset E_{\alpha,\delta} \supset \operatorname{supp}(\psi_\alpha)$. We then extend the definition of each V_α to

$$\operatorname{cone}(E_{\alpha,\delta}) := \{tu \in \mathbb{R}^n \setminus \{0\} : t > 0,\ u \in E_{\alpha,\delta}\}$$

using the homogeneity property, call this functions $\widetilde{V_\alpha}$, and show that $\widetilde{V_\alpha}$ fulfills the properties of Theorem 1 on the set $\operatorname{cone}(\operatorname{supp}(\psi_\alpha))$. From this it then follows that $V_\varepsilon(x) := \sum_{\alpha\in\mathcal{A}} \psi_\alpha(x/\|x\|)\widetilde{V}_\alpha(x)$ fulfills the promised properties of Theorem 1 globally.

Atlas and partition of unity:
Following Chapter 2 in Lee (2013), let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ be a smooth atlas for the manifold S^{n-1} with the standard smooth structure, see e.g. Examples 1.4 and 1.31 in Lee (2013). Since S^{n-1} is compact we may and will assume that the atlas is finite, i.e. $|\mathcal{A}| < \infty$. Let $(\psi_\alpha)_{\alpha\in\mathcal{A}}$ be a smooth partition of unity subordinate to the open cover $(U_\alpha)_{\alpha\in\mathcal{A}}$ of S^{n-1}. This implies that the smooth functions $\psi_\alpha: S^{n-1} \to [0, 1]$, $\alpha \in \mathcal{A}$, fulfill $\operatorname{supp}(\psi_\alpha) \subset U_\alpha$ for all $\alpha \in \mathcal{A}$ and

$$\sum_{\alpha\in\mathcal{A}} \psi_\alpha(x) = 1 \quad \text{for all } x \in S^{n-1}.$$

In particular, for every $x \in S^{n-1}$ there is an $\alpha \in \mathcal{A}$ such that $\psi_\alpha(x) > 0$ and $\operatorname{supp}(\psi_\alpha)$ is a compact subset of the open set U_α for every $\alpha \in \mathcal{A}$.

Remark 2. Let us compare our notation to the one used in Munkres (1991). Our atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ corresponds to the coordinate patches α in sec. 23. The only difference is that we consider mappings ϕ_α from S^{n-1} into $\mathbb{R}^{n-1}$, and not mappings α from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$, whose codomains cover S^{n-1}. Hence, our ϕ_α correspond to the functions α^{-1} in the first definition in sec. 23.

Our partition of unity subordinate to the open cover $(U_\alpha)_{\alpha\in\mathcal{A}}$ corresponds to the partition of unity $\phi_i: \mathbb{R}^n \to \mathbb{R}$ in Lemma 25.2 in Munkres (1991), dominated by the coordinate patches (α_i); our ψ_α are the restrictions $\phi_i\big|_{S^{n-1}}$. In particular, we may assume that

$$\psi_\alpha = \overline{\psi}_\alpha\big|_{S^{n-1}}, \quad \text{where } \overline{\psi}_\alpha \in C^\infty(\mathbb{R}^n), \tag{6}$$

for all $\alpha \in \mathcal{A}$.

Mollifier:
For every $\delta > 0$ let $\rho_\delta: \mathbb{R}^n \to \mathbb{R}$ be the smooth function

$$\rho_\delta(x) = \begin{cases} C_\delta \exp\left(\dfrac{-1}{1 - (\|x\|/\delta)^2}\right), & \text{if } \|x\| < \delta, \\ 0, & \text{otherwise,} \end{cases}$$

where the constant C_δ is chosen such that for an $x \in S^{n-1}$

$$\int_{S^{n-1}} \rho_\delta(x - y)dy = 1. \tag{7}$$

Then $\operatorname{supp}(\rho_\delta) = \overline{B_\delta}$, where $B_\delta := \{x \in \mathbb{R}^n : \|x\| < \delta\}$ and $\overline{B_\delta}$ denotes the closure of B_δ, and because of rotational symmetry (7) holds true for every $x \in S^{n-1}$.

Remark 3. Our function ρ_δ is obtained from the function $f(x) = e^{-1/x}$ for $x > 0$ and $f(x) = 0$ if $x \le 0$ in Lemma 16.1 in Munkres (1991) through $\rho_\delta(x) := C_\delta f(1 - (x/\delta)^2)$.

Remark 4. Recall that for every C^1 diffeomorphism ϕ from an open set $U \subset S^{n-1}$ to $\phi(U) \subset \mathbb{R}^{n-1}$ and any continuous function $f: S^{n-1} \to \mathbb{R}$ with $\operatorname{supp}(f) \subset U$, we have

$$\int_{S^{n-1}} f(y)dy := \int_{\phi(U)} f(\phi^{-1}(z))\Delta_{\phi^{-1}}(z)dz, \tag{8}$$

where

$$\Delta_g(z) := \sqrt{\det([Dg(z)]^T Dg(z))} \quad \text{for } g \in C^1 \tag{9}$$

and $Dg(z)$ denotes the Jacobian matrix of g at z and $[Dg(z)]^T$ its transpose. This definition is independent of the choice of the diffeomorphism ϕ; see e.g. sec. 25, in particular Lemma 25.1, in Munkres (1991).

It easily follows, that for a C^1 diffeomorphism ϕ from U to $\phi(U) \subset \mathbb{R}^{n-1}$, where $(x+B_\delta)\cap S^{n-1} \subset U \subset S^{n-1}$, we have

$$\int_{S^{n-1}} \rho_\delta(x-y)dy = \int_{\phi((x+B_\delta)\cap S^{n-1})} \rho_\delta(x-\phi^{-1}(z))\Delta_{\phi^{-1}}(z)dz.$$

Further, the definition (8) is applicable and independent of the choice of the diffeomorphism as long as $f\circ\phi^{-1}$ is Lebesgue integrable; see e.g. Theorem 19.4 in Bauer (2001).

Fix $\delta > 0$:
We will smooth out V on S^{n-1} by using convolution with the mollifier ρ_δ. Let $L > 0$ be a global Lipschitz constant for V, see Remark 1, and define

$$\delta := \min\left\{\frac{\varepsilon}{L}, \frac{\varepsilon}{3LG+c}, \frac{1}{2}\min_{\alpha\in\mathcal{A}} \mathrm{dist}(U_\alpha, \mathrm{supp}(\psi_\alpha))\right\}, \tag{10}$$

where $\mathrm{dist}(A,B) := \inf_{x\in A, y\in B}\|x-y\|$ for $A, B \subset \mathbb{R}^n$. Then, for every $\alpha\in\mathcal{A}$ and every $x \in \mathrm{supp}(\psi_\alpha)$, we have

$$(x+B_{2\delta})\cap S^{n-1} \subset U_\alpha,$$

from which, with

$$E_{\alpha,\delta} := \{x\in S^{n-1}\colon\ \mathrm{dist}(\{x\}, \mathrm{supp}(\psi_\alpha)) < \delta\},$$

it follows that

$$x \in E_{\alpha,\delta} \ \text{ implies } \ (x+B_\delta)\cap S^{n-1}\subset U_\alpha. \tag{11}$$

Smooth local approximations V_α to V on S^{n-1}:
We define smooth approximations V_α to V on $E_{\alpha,\delta}$. For $x \in E_{\alpha,\delta}$ define

$$\varphi_{\alpha,x}\colon x - E_{\alpha,\delta} \to \phi_\alpha(E_{\alpha,\delta}), \quad \varphi_{\alpha,x}(y) := \phi_\alpha(x-y),$$

i.e.

$$\varphi_{\alpha,x}^{-1}\colon \phi_\alpha(E_{\alpha,\delta}) \to x - E_{\alpha,\delta}, \quad \varphi_{\alpha,x}^{-1}(z) = x - \phi_\alpha^{-1}(z).$$

Now define the functions $V_\alpha\colon E_{\alpha,\delta}\to\mathbb{R}$ through

$$V_\alpha(x) := \int_{\phi_\alpha(E_{\alpha,\delta})} V\circ\phi_\alpha^{-1}(z)\rho_\delta(x-\phi_\alpha^{-1}(z))\Delta_{\phi_\alpha^{-1}}(z)dz \tag{12}$$

$$= \int_{\phi_\alpha(E_{\alpha,\delta})} V(x-\varphi_{\alpha,x}^{-1}(z))\rho_\delta\circ\varphi_{\alpha,x}^{-1}(z)\Delta_{\varphi_{\alpha,x}^{-1}}(z)dz,$$

where we used that $\Delta_{\phi_\alpha^{-1}}(z) = \Delta_{\varphi_{\alpha,x}^{-1}}(z)$, which is easily seen from formula (9). Note that, because of (11) and Remark 4, that

$$x\in E_{\alpha,\delta}\cap E_{\beta,\delta} \ \text{ implies } \ V_\alpha(x) = V_\beta(x), \tag{13}$$

and that for every $x\in E_{\alpha,\delta}$ we have

$$\begin{aligned} &|V_\alpha(x) - V(x)| \\ &\leq \int_{\phi_\alpha(E_{\alpha,\delta})} |V(x-\varphi_{\alpha,x}^{-1}(z)) - V(x)|\rho_\delta\circ\varphi_{\alpha,x}^{-1}(z)\Delta_{\varphi_{\alpha,x}^{-1}}(z)dz \\ &\leq \int_{\phi_\alpha(E_{\alpha,\delta})} L\delta\cdot\rho_\delta\circ\varphi_{\alpha,x}^{-1}(z)\Delta_{\varphi_{\alpha,x}^{-1}}(z)dz \\ &= L\delta\int_{S^{n-1}}\rho_\delta(x-y)dy \leq \varepsilon \end{aligned} \tag{14}$$

because $\delta \leq \varepsilon/L$ by (10).

We now extend the definition domains of the V_α forcing the homogenous property on $\widetilde{V}_\alpha$. Later we show that the orbital derivatives of the $\widetilde{V}_\alpha$ approximate the orbital derivatives of V because both $\widetilde{V}_\alpha$ and V are homogenous.

Smooth local approximations $\widetilde{V}_\alpha$ to V on $\mathbb{R}^n$:
With

$$\widetilde{\phi}_\alpha\colon\ \mathrm{cone}(U_\alpha)\to\mathbb{R}^n, \quad \widetilde{\phi}_\alpha(x) := \big(\|x\|, \phi_\alpha(x/\|x\|)\big),$$

consider the atlas $\{(\mathrm{cone}(U_\alpha), \widetilde{\phi}_\alpha)\}_{\alpha\in\mathcal{A}}$ for the manifold $\mathbb{R}^n\setminus\{0\}$. We define the functions $\widetilde{V}_\alpha\colon\mathbb{R}^n\to\mathbb{R}$ through

$$\widetilde{V}_\alpha\circ\widetilde{\phi}_\alpha^{-1}(t,u) = tV_\alpha(u) \ \text{ for } \ (t,u)\in\widetilde{\phi}_\alpha(\mathrm{cone}(E_{\alpha,\delta})),$$

i.e.

$$\widetilde{V}_\alpha(x) := \|x\|V_\alpha(x/\|x\|) \ \text{ for } x\in\mathrm{cone}(E_{\alpha,\delta}), \tag{15}$$

and we set $\widetilde{V}_\alpha(x) = 0$ otherwise.

That $\widetilde{V}_\alpha$ is C^∞ on $\mathrm{cone}(E_{\alpha,\delta})$ is easily seen from the definition of $\widetilde{V}_\alpha$ and formula (12), i.e.

$$\widetilde{V}_\alpha(x) = \|x\|\int_{\phi_\alpha(E_{\alpha,\delta})} V\circ\phi_\alpha^{-1}(z)\rho_\delta\left(\frac{x}{\|x\|} - \phi_\alpha^{-1}(z)\right)\Delta_{\phi_\alpha^{-1}}(z)dz.$$

Because the integrand is continuous and compactly supported, it follows from Lebesgue's dominated convergence theorem that we can differentiate w.r.t. x under the integral, see e.g. Corollary 16.3 in Bauer (2001), and it is straightforward to use induction to see that since ρ_δ is C^∞, so is $\widetilde{V}_\alpha$ on $\mathrm{cone}(E_{\alpha,\delta})$.

Orbital derivatives of $\widetilde{V}_\alpha$:
We now show that assumption (16), i.e.

$$\limsup_{h\to0+}\frac{V(x_h^i)-V(x)}{h} \leq -c\|x\| \tag{16}$$

for all $x\in\mathbb{R}^n\setminus\{0\}$ and $i = 1,2,\ldots,N$, implies

$$\limsup_{h\to0+}\frac{\widetilde{V}_\alpha(x_h^i)-\widetilde{V}_\alpha(x)}{h} \leq -(c-\varepsilon)\|x\| \tag{17}$$

for every $\alpha\in\mathcal{A}$, $x\in\mathrm{cone}(\mathrm{supp}(\psi_\alpha))$, and $i=1,2,\ldots,N$. Since $\widetilde{V}_\alpha$ is C^∞ on $\mathrm{cone}(E_{\alpha,\delta})\supset\mathrm{cone}(\mathrm{supp}(\psi_\alpha))$, (17) is equivalent to

$$\nabla\widetilde{V}_\alpha(x_i)g_i(x) \leq -(c-\varepsilon)\|x\|. \tag{18}$$

For the proof let $\alpha\in\mathcal{A}$, $x\in\mathrm{cone}(\mathrm{supp}(\psi_\alpha))$, and $i\in\{1,2,\ldots,N\}$ be fixed, but arbitrary. Then there is a $\delta^* > 0$ such that $x+\mathcal{B}_{\delta^*}\subset\mathrm{cone}(U_\alpha)$. Further, since $x\neq0$, we have

$$\begin{aligned} 0 < c\|x\| &\leq \left|\limsup_{h\to0+}\frac{V(x_h^i)-V(x)}{h}\right| \\ &\leq L\left|\limsup_{h\to0+}\left\|\frac{x_h^i-x}{h}\right\|\right| \leq L\|g_i(x)\|, \end{aligned}$$

i.e. $g_i(x)\neq0$. Let $h^*>0$ be so small that

$$x_h^i\in\mathrm{cone}(E_{\alpha,\delta}) \ \text{ and } \ \left\|\frac{x_h^i-x}{h}\right\| \leq 2\|g_i(x)\|$$

for all $0<h\leq h^*$. For such h we have

$$\frac{\widetilde{V}_\alpha(x_h^i)-\widetilde{V}_\alpha(x)}{h} = \frac{\|x_h^i\|V_\alpha\left(\frac{x_h^i}{\|x_h^i\|}\right)-\|x\|V_\alpha\left(\frac{x}{\|x\|}\right)}{h} \tag{19}$$

$$= \frac{1}{h}\int_{\phi_\alpha(E_{\alpha,\delta})}\left[\|x_h^i\|V\left(\frac{x_h^i}{\|x_h^i\|}-\varphi_{\alpha,x}^{-1}(z)\right)\right.$$
$$\left.-\|x\|V\left(\frac{x}{\|x\|}-\varphi_{\alpha,x}^{-1}(z)\right)\right]\rho_\varepsilon\circ\varphi_{\alpha,x}^{-1}(z)\Delta_{\varphi_{\alpha,x}^{-1}}(z)dz$$

$$= \int_{\phi_\alpha(E_{\alpha,\delta})}\frac{1}{h}\left[\|x_h^i\|V\left(\frac{x_h^i}{\|x_h^i\|}-\varphi_{\alpha,x}^{-1}(z)\right)\right.$$
$$\left.-\|x\|V\left(\frac{x_h^i}{\|x\|}-\varphi_{\alpha,x}^{-1}(z)\right)\right]\rho_\varepsilon\circ\varphi_{\alpha,x}^{-1}(z)\Delta_{\varphi_{\alpha,x}^{-1}}(z)dz$$

$$+\int_{\phi_\alpha(E_{\alpha,\delta})}\frac{\|x\|}{h}\left[V\left(\frac{x_h^i}{\|x\|}-\varphi_{\alpha,x}^{-1}(z)\right)\right.$$
$$\left.-V\left(\frac{x}{\|x\|}-\varphi_{\alpha,x}^{-1}(z)\right)\right]\rho_\varepsilon\circ\varphi_{\alpha,x}^{-1}(z)\Delta_{\varphi_{\alpha,x}^{-1}}(z)dz.$$

We now show that the absolute values of the integrands in both integrals on the right-hand-side of (19) are dominated by integrable functions. Hence, we can use Fatou's lemma, i.e. with g and f_n integrable, $|f_n|\le g$ for $n\in\mathbb{N}$, we have

$$\int\liminf_{n\to\infty}(g-f_n)\le\liminf_{n\to\infty}\int(g-f_n),$$

i.e.

$$\limsup_{n\to\infty}\int f_n\le\int\limsup_{n\to\infty}f_n$$

because $\limsup_{n\to\infty}f_n=-\liminf_{n\to\infty}(-f_n)$; see e.g. Lemma 15.2 in Bauer (2001).

For the integrand of the first integral we have with $y=\varphi_{\alpha,x}^{-1}(z)$ that $\|y\|<\delta$ and with $G>0$ from (1) we get

$$\frac{1}{h}\left|\|x_h^i\|V\left(\frac{x_h^i}{\|x_h^i\|}-y\right)-\|x\|V\left(\frac{x_h^i}{\|x\|}-y\right)\right| \tag{20}$$
$$=\frac{1}{h}\left|V\left(x_h^i-\|x_h^i\|y\right)-V\left(x_h^i-\|x\|y\right)\right|$$
$$\le\frac{L}{h}\left\|(\|x\|-\|x_h^i\|)y\right\|$$
$$\le L\left\|\frac{x_h^i-x}{h}\right\|\|y\|$$
$$\le 2L\|g_i(x)\|\delta$$
$$\le 2LG\delta\|x\|.$$

By the estimate (20) we additionally see that the absolute value of the limes superior of the first integral on the right-hand-side of (19) is upper bounded by $2LG\delta\|x\|$ as $h\to 0+$.

For the integrand of the second integral we get

$$\frac{\|x\|}{h}\left|V\left(\frac{x_h^i}{\|x\|}-y\right)-V\left(\frac{x}{\|x\|}-y\right)\right| \tag{21}$$
$$=\frac{1}{h}\left|V\left(x_h^i-\|x\|y\right)-V\left(x-\|x\|y\right)\right|$$
$$\le\frac{L}{h}\left\|x_h^i-x\right\|$$
$$\le 2L\|g_i(x)\|.$$

Together (20) and (21) establish that we can use Fatou's lemma and as $h\to 0+$ the limes superior of the left-hand-side of (19) is upper bounded by the integrals of the limes superior of the integrands on the right-hand-side.

We have already seen that the first integral in (19) is upper bounded by $2LG\delta\|x\|$ as $h\to 0+$. For the second integral, first note that

$$\frac{\|x\|}{h}\left[V\left(\frac{x_h^i}{\|x\|}-y\right)-V\left(\frac{x}{\|x\|}-y\right)\right] \tag{22}$$
$$=\frac{V\left(x_h^i-\|x\|y\right)-V\left([x-\|x\|y]_h^i\right)}{h}$$
$$+\frac{V\left([x-\|x\|y]_h^i\right)-V\left(x-\|x\|y\right)}{h},$$

where $[x-\|x\|y]_h^i$ is the trajectory (3) for the initial vector $x-\|x\|y$. For the first term on the right-hand-side of (22), we have the upper bound

$$\left|\frac{V\left(x_h^i-\|x\|y\right)-V\left([x-\|x\|y]_h^i\right)}{h}\right|$$
$$\le L\left\|\frac{x_h^i-\|x\|y-[x-\|x\|y]_h^i}{h}\right\|$$
$$\le L\left\|\frac{x_h^i-x}{h}-\frac{[x-\|x\|y]_h^i-(x-\|x\|y)}{h}\right\|$$

and therefore

$$\limsup_{h\to 0+}\left|\frac{V\left(x_h^i-\|x\|y\right)-V\left([x-\|x\|y]_h^i\right)}{h}\right|$$
$$\le L\left\|g_i(x)-g_i(x-\|x\|y)\right\|$$
$$\le LG\|x\|\|y\|$$
$$\le LG\delta\|x\|.$$

For the second term on the right-hand-side of (22) we have by the assumption (16), that

$$\limsup_{h\to 0+}\frac{V\left([x-\|x\|y]_h^i\right)-V\left(x-\|x\|y\right)}{h}\le -c\|x-\|x\|y\|$$
$$\le -c\left(\|x\|-\|\|x\|y\|\right)\le -c\|x\|+\delta c\|x\|.$$

Hence, putting the pieces together, delivers

$$\limsup_{h\to 0+}\frac{\widetilde{V}_\alpha(x_h^i)-\widetilde{V}_\alpha(x)}{h}\le 3LG\delta\|x\|-c\|x\|+\delta c\|x\|$$
$$\le -(c-\varepsilon)\|x\|,$$

because $(3LG+c)\delta\le\varepsilon$ by (10). Since $\alpha\in\mathcal{A}$, $x\in\mathrm{cone}(\mathrm{supp}(\psi_\alpha))$, and $i\in\{1,2,\dots,N\}$ were arbitrary, we have shown (17).

The function V_ε and its properties:
We define the function $V_\varepsilon\colon\mathbb{R}^n\to\mathbb{R}$ by

$$V_\varepsilon(x)=\sum_{\alpha\in\mathcal{A}}\widetilde{\psi}_\alpha(x)\widetilde{V}_\alpha(x),$$

where

$$\widetilde{\psi}_\alpha(x):=\begin{cases}\psi_\alpha(x/\|x\|), & \text{for } x\in\mathbb{R}^n\setminus\{0\},\\ 0, & \text{for } x=0.\end{cases}$$

We now show all the properties of V_ε stated in Theorem 1
1) $V_\varepsilon\in C^\infty(\mathbb{R}^n\setminus\{0\})$:
Note that for every $\alpha\in\mathcal{A}$ the function $\widetilde{\psi}_\alpha=\overline{\psi}_\alpha\circ g$,

$g(x) = x/\|x\|$, is the composition of C^∞ functions, see (6), and is therefore C^∞ on $\mathbb{R}^n \setminus \{0\}$. Further, $\mathrm{supp}(\widetilde{\psi}_\alpha) = \mathrm{cone}(\mathrm{supp}(\psi_\alpha)) \cup \{0\}$. Since $\widetilde{V}_\alpha$ is C^∞ on the open set $\mathrm{cone}(E_{\alpha,\delta}) \supset \mathrm{cone}(\mathrm{supp}(\psi_\alpha))$, i.e. $x \notin \mathrm{cone}(E_{\alpha,\delta})$ implies $x = 0$ or $x \in \mathbb{R}^n \setminus \mathrm{supp}(\widetilde{\psi}_\alpha)$, the function $x \mapsto \widetilde{\psi}_\alpha(x)\widetilde{V}_\alpha(x)$ is C^∞ on $\mathbb{R}^n \setminus \{0\}$. Hence, V_ε is C^∞ on $\mathbb{R}^n \setminus \{0\}$.

2) $\nabla V_\varepsilon(x) g_i(x) \le -(c-\varepsilon)\|x\|$:
For any $j \in \{1,2,\ldots,n\}$ and for every $x \neq 0$ we have, with

$$\mathcal{A}_x := \{\alpha \in \mathcal{A} \colon x \in \mathrm{supp}(\widetilde{\psi}_\alpha)\},$$

that

$$\begin{aligned}\frac{\partial V_\varepsilon}{\partial x_j}(x) &= \sum_{\alpha\in\mathcal{A}_x}\left[\frac{\partial\widetilde{\psi}_\alpha}{\partial x_j}(x)\widetilde{V}_\alpha(x) + \widetilde{\psi}_\alpha(x)\frac{\partial\widetilde{V}_\alpha}{\partial x_j}(x)\right]\\ &= \sum_{\alpha\in\mathcal{A}_x}\widetilde{\psi}_\alpha(x)\frac{\partial\widetilde{V}_\alpha}{\partial x_j}(x)\end{aligned}$$

because $\alpha, \beta \in \mathcal{A}_x$ implies $\widetilde{V}_\alpha(x) = \widetilde{V}_\beta(x)$, see (13) and (15), and for a particular $\beta \in \mathcal{A}_x$ we have

$$\sum_{\alpha\in\mathcal{A}_x}\frac{\partial\widetilde{\psi}_\alpha}{\partial x_j}(x)\widetilde{V}_\alpha(x) = \widetilde{V}_\beta(x)\frac{\partial}{\partial x_j}\underbrace{\left(\sum_{\alpha\in\mathcal{A}_x}\widetilde{\psi}_\alpha(x)\right)}_{=1} = 0.$$

Hence, from (18) it follows for every $x \neq 0$ and $i = 1,2,\ldots,N$, that

$$\begin{aligned}\nabla V_\varepsilon(x) g_i(x) &= \sum_{\alpha\in\mathcal{A}_x}\widetilde{\psi}_\alpha(x)\nabla\widetilde{V}_\alpha(x) g_i(x)\\ &\le \sum_{\alpha\in\mathcal{A}_x}\widetilde{\psi}_\alpha(x)[-(c-\varepsilon)\|x\|]\\ &\le -(c-\varepsilon)\|x\|.\end{aligned}$$

3) Homogeneity of V_ε:
For $x \in \mathbb{R}^n \setminus \{0\}$, $s > 0$, and $\beta \in \mathcal{A}_x$, we have by (15) that

$$\begin{aligned}V_\varepsilon(sx) &= \widetilde{V}_\beta(sx)\sum_{\alpha\in\mathcal{A}_x}\widetilde{\psi}_\alpha(sx)\\ &= \|sx\| V_\beta\left(\frac{sx}{\|sx\|}\right)\sum_{\alpha\in\mathcal{A}_x}\widetilde{\psi}_\alpha(sx)\\ &= s\|x\| V_\beta\left(\frac{x}{\|x\|}\right)\sum_{\alpha\in\mathcal{A}_x}\widetilde{\psi}_\alpha(x)\\ &= s\sum_{\alpha\in\mathcal{A}_x}\widetilde{\psi}_\alpha(x)\widetilde{V}_\alpha(x)\\ &= sV_\varepsilon(x).\end{aligned}$$

Hence, V_ε is homogenous of order one.

4) $|V_\varepsilon(x) - V(x)| \le \varepsilon\|x\|$:
For $x = 0$ the estimate is clear and for $x \in \mathbb{R}^n \setminus \{0\}$ we have by (15) and (14), that

$$\begin{aligned}|V_\varepsilon(x) - V(x)| &= \sum_{\alpha\in\mathcal{A}}\widetilde{\psi}_\alpha(x)|\widetilde{V}_\alpha(x) - V(x)|\\ &= \sum_{\alpha\in\mathcal{A}}\psi_\alpha\left(\frac{x}{\|x\|}\right)\|x\|\left|\widetilde{V}_\alpha\left(\frac{x}{\|x\|}\right) - V\left(\frac{x}{\|x\|}\right)\right|\\ &\le \varepsilon\|x\|.\end{aligned}$$

5) $V(x) = V(-x) \Rightarrow V_\varepsilon(x) = V_\varepsilon(-x)$:
Now assume that $V(x) = V(-x)$ for all $x \in \mathbb{R}^n$. Because $S^{n-1} = -S^{n-1}$ and $\rho_\delta(z) = \rho_\delta(-z)$ for all $z \in \mathbb{R}^n$, this follows from

$$\begin{aligned}V_\varepsilon(x) &= \int_{S^{n-1}} V(y)\rho_\delta(x-y)dy = \int_{S^{n-1}} V(-z)\rho_\delta(x+z)dz\\ &= \int_{S^{n-1}} V(z)\rho_\delta(-x-z)dz = V_\varepsilon(-x)\end{aligned}$$

using the coordinate transform $y = -z$. In more detail, let $\alpha, \beta \in \mathcal{A}$ be such that

$$x/\|x\| \in \mathrm{supp}(\psi_\alpha) \quad \text{and} \quad -x/\|-x\| \in \mathrm{supp}(\psi_\beta)$$

and define

$$[\phi_\alpha^-] \colon -U_\alpha \to \phi_\alpha(U_\alpha), \quad [\phi_\alpha^-](y) = \phi_\alpha(-y),$$

i.e. $[\phi_\alpha^-]^{-1}(z) = -\phi_\alpha^{-1}(z)$. From the formula (9) we see that $\Delta_{[\phi_\alpha^-]^{-1}}(z) = \Delta_{\phi_\alpha^{-1}}(z)$ and

$$\begin{aligned}V_\alpha(x) &= \int_{\phi_\alpha(E_{\alpha,\delta})} V\circ\phi_\alpha^{-1}(z)\rho_\delta(x - \phi_\alpha^{-1}(z))\Delta_{\phi_\alpha^{-1}}(z)dz\\ &= \int_{\phi_\alpha(E_{\alpha,\delta})} V\circ[\phi_\alpha^-]^{-1}(z)\rho_\delta(x + [\phi_\alpha^-]^{-1}(z))\Delta_{[\phi_\alpha^-]^{-1}}(z)dz\\ &= \int_{\phi_\alpha(E_{\alpha,\delta})} V\circ[\phi_\alpha^-]^{-1}(z)\rho_\delta(-x - [\phi_\alpha^-]^{-1}(z))\Delta_{[\phi_\alpha^-]^{-1}}(z)dz\\ &= \int_{\phi_\beta(E_{\alpha,\delta})} V\circ\phi_\beta^{-1}(z)\rho_\delta(x - \phi_\beta^{-1}(z))\Delta_{\phi_\beta^{-1}}(z)dz\\ &= V_\beta(-x)\end{aligned}$$

by Remark 4 and from (13) it follows that

$$\begin{aligned}V_\varepsilon(x) &= \widetilde{V}_\alpha(x)\sum_{\alpha'\in\mathcal{A}}\widetilde{\psi}_{\alpha'}(x)\\ &= \widetilde{V}_\beta(-x)\sum_{\beta'\in\mathcal{A}}\widetilde{\psi}_{\beta'}(x)\\ &= V_\varepsilon(-x).\end{aligned}$$

4. CONCLUSIONS

We proved, using elementary differential geometry, that the existence of a locally Lipschitz continuous, homogenous of order one Lyapunov function $V \colon \mathbb{R}^n \to \mathbb{R}$ for a switched system $\dot{x} = g_\sigma(x)$, $\sigma \colon [0,\infty) \to \{1,2,\ldots,N\}$ and the g_i are globally Lipschitz continuous, implies the existence of a smooth, except at the origin, homogenous of order one Lyapunov function V_ε for the system. Further, we established that V_ε can approximate V and its orbital derivatives arbitrary close; this is important when studying numerical methods to compute Lyapunov functions. Finally, we showed that our smoothing procedure preserves symmetry of the Lyapunov function.

REFERENCES

Andersen, A., Giesl, P., and Hafstein, S. (2023a). Common Lyapunov functions for switched linear systems: Linear programming based approach. *IEEE Contr. Syst. Lett.*, 7, 901–906.

Andersen, S., August, E., Hafstein, S., and Piccini, J. (2023b). Lyapunov function computation for linear switched systems: Comparison of SDP and LP approaches. In *Proceedings of the 13th International Conference on Simulation and Modeling Methodologies, Technologies and Applications*, volume 1, 61–70.

Bauer, H. (2001). *Measure and Integration Theory*. Walter de Gruyter, 1st edition.

Davrazos, G. and Koussoulas, N. (2001). A review of stability results for switched and hybrid systems. In *Proceedings of 9th Mediterranean Conference on Control and Automation*. Dubrovnik, Croatia.

Dayawansa, W. and Martin, C. (1999). A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Tra*, (44), 751–760.

Della Rossa, M., Tanwani, A., and Zaccarian, L. (2020). Max-min Lyapunov functions for switched systems and related differential inclusions. *Automatica*, 120, Article 109123.

Haar, A. (1933). Der Massbegriff in der Theorie der kontinuierlichen Gruppen. *Annals of Mathematics*, 34(1), 147–169.

Hafstein, S. (2023). Lyapunov functions for switched linear systems: Proof of convergence for an LP computational approach. *IEEE Control Systems Letters*, 7, 3283–3288. doi:10.1109/LCSYS.2022.3228857.

Hafstein, S. and Tanwani, A. (2023). Linear programming based lower bounds on average dwell-time via multiple Lyapunov functions. *Eur. J. Control*, 74, art.icle nr. 100838.

Lee, J. (2013). *Introduction to Smooth Manifolds*. Springer, 2nd edition.

Liberzon, D. (2003). *Switching in systems and control*. Systems & Control: Foundations & Applications. Birkhäuser.

Munkres, J. (1991). *Analysis on Manifolds*. CRC Press, Boca Raton, FL, USA.

Polanski, A. (1997). Lyapunov functions construction by linear programming. *IEEE Trans. Automat. Control*, 42(7), 1113–1116.

Sun, Z. and Ge, S. (2011). *Stability Theory of Switched Dynamical Systems*. Communications and Control Engineering. Springer.