Minimum Average Dwell-Time Computations via Continuous Piecewise Quadratic Lyapunov Functions

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***Abstract*— Recently, continuous, piecewise linear Lyapunov functions were used to compute the minimum average-dwell time needed to assert global exponential stability for switched linear systems. Here, we use continuous, piecewise quadratic Lyapunov functions to do the same and compare the efficiency of the two methods. As in the case for piecewise linear Lyapunov functions, we use linear programming to parameterize our Lyapunov functions on a conical subdivision of the state-space.**

I. Introduction

We consider switched linear systems of the form

$$\dot{\mathbf{x}} = A_\sigma \mathbf{x}, \quad \sigma(t) \in \{1, 2, \dots, N_{\text{subs}}\} =: \mathcal{P}, \tag{1}$$

where the $A_i \in \mathbb{R}^{n\times n}$ are Hurwitz matrices, i.e. the real-parts of the eigenvalues are negative and the origin is globally exponentially stable (GES) for the individual subsystems $\dot{\mathbf{x}} = A_i\mathbf{x}$. Such systems have been intensively studied in the literature, see e.g. [9], [21], [19], [6], [8], [7]. GES of the origin for the arbitrary switched system (1) means that there exist constants $M, \beta > 0$, such that for any switching signal $\sigma\colon \mathbb{R}_+ \to \mathcal{P}$, $\mathbb{R}_+ := [0,\infty)$, i.e. right-continuous and only with finitely many discontinuity points on any finite interval, and any initial value $\boldsymbol{\xi} \in \mathbb{R}^n$, we have

$$\|\mathbf{x}_\sigma(t;\boldsymbol{\xi})\|_2 \le M \exp(-\beta t)\|\boldsymbol{\xi}\|_2$$

for all $t \in \mathbb{R}_+$. Here $t \mapsto \mathbf{x}_\sigma(t;\boldsymbol{\xi})$, $\mathbf{x}_\sigma(0;\boldsymbol{\xi}) = \boldsymbol{\xi}$, is the solution to (1) with switching σ. This means that $t \mapsto \mathbf{x}_\sigma(t;\boldsymbol{\xi})$ is continuous and with $0 = t_0 < t_1 < t_2 < \dots$, where the $t_1, t_2, \dots$ are the discontinuity points of σ, also called switching times, we have $\dot{\mathbf{x}}(t;\boldsymbol{\xi}) = A_{\sigma(t_i)}\mathbf{x}(t;\boldsymbol{\xi})$ for all $t_i < t < t_{i+1}$, $i = 0, 1, \dots$.

Although the origin is GES for each subsystem $\dot{\mathbf{x}} = A_i\mathbf{x}$, $i \in \mathcal{P}$, this does not necessarily hold true for the arbitrary switched system, see e.g. [18] for a counterexample. However, if one demands a large enough minimum average-dwell time (MADT) on the switching, then the origin is always GES for the switched system; see e.g. [15], [18]. Denote by $N_\sigma(t,s)$ the number of switching times of the switching signal $\sigma\colon \mathbb{R}_+ \to \mathcal{P}$ on the open time-interval (s,t). The switching signal σ is said to have an MADT $\tau_a > 0$, if there exists a constant $N_0 > 0$ such that

$$N_\sigma(t,s) \le N_0 + \frac{t-s}{\tau_a}$$

for every finite interval $(s,t) \subset \mathbb{R}_+$. If there are no restrictions on $N_\sigma(t,s)$, except that it is finite for any finite interval (s,t), we say the MADT is equal to zero. We denote the set of all switching signals with MADT $\tau_a \ge 0$ with Σ_{τ_a}.

An interesting question for the system (1) with concrete matrices A_i, $i \in \mathcal{P}$, is now: what is the MADT $\tau_a \ge 0$, for which we can assert the GES of the origin for all switchings in Σ_{τ_a}? To tackle this question, the following theorem from [4], which is adapted from Theorem 3.2 in [18], is useful. It gives a sufficient condition in terms of Lyapunov functions for the individual subsystems. Recall that $\mathcal{K}_\infty$ is the set of strictly increasing, unbounded, continuous functions $\beta\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\beta(0) = 0$.

Theorem 1: Assume that for each $i \in \mathcal{P}$, there exists a locally Lipschitz continuous function $V_i\colon \mathbb{R}^n \to \mathbb{R}$, such that

$$\underline{a}^*(\|\mathbf{x}\|_2) \le V_i(\mathbf{x}) \le \overline{a}^*(\|\mathbf{x}\|_2), \quad \forall i \in \mathcal{P}, \tag{2a}$$

$$D^+V_i(\mathbf{x}, A_i\mathbf{x}) \le -\alpha V_i(\mathbf{x}), \quad \forall i \in \mathcal{P}, \tag{2b}$$

$$V_i(\mathbf{x}) \le \mu V_j(\mathbf{x}), \quad \forall i,j \in \mathcal{P}, \tag{2c}$$

where $\overline{a}^*, \underline{a}^* \in \mathcal{K}_\infty$, $\alpha > 0$, and $\mu \ge 1$. Then the origin is GES for the switched system (1) for every switching σ with average dwell-time τ fulfilling

$$\tau > \frac{\ln(\mu)}{\alpha}, \tag{3}$$

i.e. for every $\sigma \in \Sigma_\tau$ if $\mu > 1$ and for an arbitrary switching σ if $\mu = 1$.

□

Here,

$$D^+V_i(\mathbf{x}, A_i\mathbf{x}) := \limsup_{h\to 0+} \frac{V_i(\mathbf{x} + hA_i\mathbf{x}) - V_i(\mathbf{x})}{h}$$

is the Dini-derivative of V_i along the solution trajectory of the system $\dot{\mathbf{x}} = A_i\mathbf{x}$ at the point $\mathbf{x} \in \mathbb{R}^n$.

In [14], [2], [4] linear programming (LP) was used to parameterize continuous and piecewise affine (CPA) Lyapunov functions, that are then extended to continuous, cone-wise linear Lyapunov functions, for the system (1) to compute sufficiently large MADT to assert GES of the origin; of course one is interested in as low MADT as possible needed to assert stability. In more detail, in Corollary 2 in [14] the conditions of Theorem 1 are enforced through the stricter conditions

$$\underline{a}\|\mathbf{x}\|_2 \le V_i(\mathbf{x}) \le \overline{a}\|\mathbf{x}\|_2, \quad \forall i \in \mathcal{P}, \tag{4a}$$

$$D^+V_i(\mathbf{x}, A_i\mathbf{x}) \le -\alpha\|\mathbf{x}\|_2, \quad \forall i \in \mathcal{P}, \tag{4b}$$

$$V_i(\mathbf{x}) \le \mu V_j(\mathbf{x}), \quad \forall i,j \in \mathcal{P}, \tag{4c}$$

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for fixed constants $\underline{a}, \overline{a} > 0$ and $\mu \geq 1$, and then $\alpha > 0$ was maximized. From the existence of these Lyapunov functions one can then assert GES of the origin for every switching signal with MADT $\tau > \overline{a}\ln(\mu)/\alpha$. In [4] CPA Lyapunov functions fulfilling (4a) and (4c), but (2b) instead of (4b), were computed for constants $\underline{a}, \overline{a}, \alpha > 0$ and $\mu \geq 1$. Note that these are just the conditions in Theorem 1 for CPA Lyapunov functions. From the existence of these Lyapunov functions one can then assert GES of the origin for every switching signal with MADT $\tau > 0$ fulfilling (3).

Note that while one can maximize $\alpha > 0$ in the LP associated to (4) to obtain as low $\tau > 0$ as possible for the given constants $\underline{a}, \overline{a} > 0$ and $\mu \geq 1$, this is not possible in an LP for (2) because of the term $\alpha V_i(\mathbf{x})$, which is quadratic in the variables. Hence, in [4] an algorithm was presented for efficiently searching the (α, μ)-plane for values delivering the lowest MADT needed to assert GES and we will use this algorithm in our numerical experiments.

Note that the LP method in [14] was able to deliver lower estimates on the MADT needed to assert GES of the origin than a standard linear matrix inequality (LMI) approach to compute quadratic Lyapunov functions, see [2]. However, a triangulation of the state-space is needed and for some systems this can be limiting. Indeed, as discussed in [10], a large number of simplices in the triangulation is needed to compute CPA Lyapunov functions for some systems, in particular if the negative real parts of some eigenvalues are small but the imaginary parts are not. To tackle this problem continuous and piecewise quadratic (CPQ) Lyapunov functions have been suggested, e.g. in [17], [16], but until recently one needed semidefinite programming (SDP) to parameterize such Lyapunov functions using LMIs. However, recently in [20] an LP approach was suggested to parameterize CPQ Lyapunov functions for switched linear systems; its efficient implementation was described in [5]. In this paper we present an LP problem to parameterize CPQ Lyapunov functions for the individual subsystems $\dot{\mathbf{x}} = A_i\mathbf{x}$, $i \in \mathcal{P}$, that additionally fulfil the compatibility conditions (2c), and we compare this method with the method in [4], which uses LP to parameterize CPA Lyapunov functions, to obtain a (low) MADT needed to assert GES of the origin for the switched system (1).

The rest of this paper is organized as follows: In Section II we describe the LP problem to compute CPQ Lyapunov functions that deliver a MADT $\tau \geq 0$, such that GES of the origin is asserted for the switched system (1) for every switching $\sigma \in \Sigma_\tau$. In Section III we compare our novel approach using CPQ Lyapunov functions to the CPA Lyapunov functions approach from [4]. Since $\sigma \in \Sigma_0$ corresponds to arbitrary switching σ, this analysis includes a study of the computation of CPQ common Lyapunov functions (CLFs) and CPA CLFs. Finally, in Section IV we give some conclusions.

II. CPQ Lyapunov functions for MADT analysis

To compute CPQ Lyapunov functions fulfilling the conditions of Theorem 1 for the system (1) in order to obtain a MADT that guaranties exponential stability of the origin, we adapt the method to compute CPQ Lyapunov functions in [5], [20]. Let us recapitulate the method: first a subdivision of the state-space into cones with the apex as the origin is needed. This is done by using the triangulation $\mathcal{T}_K^{\mathbf{F}}$ described in detail in e.g. [14], and then extending the n-simplices to cones; the parameter $K \in \mathbb{N}$ fixes the number of the simplices in the triangulation $\mathcal{T}_K^{\mathbf{F}}$, which is approximately spherically symmetric [1]. See Fig. 1 for a schematic figure of the triangulation $\mathcal{T}_1^{\mathbf{F}}$ in two-dimensions.

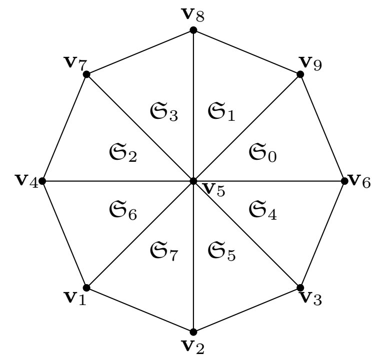

Fig. 1: Triangulation $\mathcal{T}_1^{\mathbf{F}}$ with simplices $\mathfrak{S}_\nu$ and vertices $\mathbf{v}_j$ indexed

In a little more detail, we have n-simplices

$$\mathfrak{S}_\nu := \mathrm{co}\{\mathbf{x}_0^\nu, \mathbf{x}_1^\nu, \ldots, \mathbf{x}_n^\nu\}, \quad \nu \in I,$$

where I is an index set for the simplices and $\mathbf{x}_0^\nu = \mathbf{0}$ for all $\nu \in I$. These simplices have the property that two different simplices intersect in a common face and that $\bigcup_{\nu \in I} \mathfrak{S}_\nu$ is a neighbourhood of the origin. Each point $\mathbf{x} \in \mathfrak{S}_\nu$ can be written uniquely as a convex combination of the vertices of $\mathfrak{S}_\nu$, i.e. $\mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}_i^\nu$, $\sum_{i=0}^n \lambda_i = 1$, $\lambda_i \in [0,1]$. For each simplex $\mathfrak{S}_\nu$ we define the cone (recall $\mathbf{x}_0^\nu = \mathbf{0}$)

$$\mathfrak{C}_\nu := \mathrm{cone}\{\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \ldots, \mathbf{x}_n^\nu\} = \left\{ \sum_{i=1}^n \lambda_i \mathbf{x}_i^\nu \colon \lambda_i \geq 0 \right\}$$

and these cones subdivide the state-space $\mathbb{R}^n$.

Our LP problem will attempt to parameterize Lyapunov functions $V_i \colon \mathbb{R}^n \to \mathbb{R}$ as in Theorem 1, that additionally are quadratic on each cone $\mathfrak{S}_\nu$. For this we need a parametrized representation of such CPQ functions and we need to express the conditions (2) as linear constraints in these parameters. Let us first discuss the parametrized representation.

For each simplex $\mathfrak{S}_\nu$, $\nu \in I$, define the matrix $X_\nu := [\mathbf{x}_1^\nu, \mathbf{x}_2^\nu, \cdots, \mathbf{x}_n^\nu] \in \mathbb{R}^{n\times n}$, i.e. the $\mathbf{x}_j^\nu$ are the columns of X_ν; see e.g. [3] or Remark 9 in [11] for a discussion of this definition. Clearly, every point $\mathbf{x} \in \mathfrak{C}_\nu$ can be written in a unique way as $\mathbf{x} = X_\nu \boldsymbol{\lambda}$, $\boldsymbol{\lambda} \in \mathbb{R}_+^n$. On each cone $\mathfrak{C}_\nu$ the function V_i, $i \in \mathcal{P}$, will have the formula $V_i(\mathbf{x}) = \mathbf{x}^T P_i^\nu \mathbf{x}$, $P_i^\nu \in \mathbb{R}^{n\times n}$.

Now one could proceed by using the entries of the matrices P_i^ν, $\nu \in I$ and $i \in \mathcal{P}$ as variables, to express the conditions (6) below in these variables. However, we prefer to use the parametrization from [5], [20], [16]. For this we collect all the variables for the matrices P_i^ν, $\nu \in I$, in a symmetric matrix of variables $\Phi_i \in \mathbb{R}^{p\times p}$, such that for each cone $\mathfrak{C}_\nu$, $\nu \in I$, we have $\Psi_i^\nu := X_\nu^T P_i^\nu X_\nu$, where the (k,ℓ)-th entry $\psi_{k,\ell}^{i,\nu}$ of the matrix $\Psi_i^\nu \in \mathbb{R}^{n\times n}$ is the $(g_\nu(k), g_\nu(\ell))$-th entry $\varphi^{(i)}_{g_\nu(k),g_\nu(\ell)}$ of the matrix of variables Φ_i. The functions

$$g_\nu\colon \{1,2,\ldots,n\} \to \{1,2,\ldots,p\}, \quad \nu \in I,$$

are defined such that $\mathbf{x}_i^\nu = \mathbf{v}_{g_\nu(i)}$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are the different vertices of the triangulation $\mathcal{T}_K^{\mathbf{F}}$ and take care of the correct assignment to the variables in Φ_i for P_i^ν. Note that when two vertices of different simplices $\mathfrak{S}_\nu$ and $\mathfrak{S}_\mu$ are equal, i.e. $\mathbf{x}_i^\nu = \mathbf{x}_j^\mu$, then $g_\nu(i) = g_\mu(j)$. This can be done in an identical way for all $i \in \mathcal{P}$. Now, for an $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{x}_i^\nu = X^\nu \boldsymbol{\lambda} \in \mathfrak{C}_\nu$ we have

$$\begin{aligned} V_i(\mathbf{x}) &= \mathbf{x}^T P_i^\nu \mathbf{x} = \boldsymbol{\lambda}^T X_\nu^T (X_\nu^{-1})^T \Psi_i^\nu X_\nu^{-1} X_\nu \boldsymbol{\lambda} \\ &= \boldsymbol{\lambda}^T \Psi_i^\nu \boldsymbol{\lambda} = \sum_{k,\ell=1}^n \lambda_k \lambda_\ell \varphi^{(i)}_{g_\nu(k),g_\nu(\ell)}. \end{aligned} \tag{5}$$

Hence, the $n\times n$-matrix $\Psi_i^\nu = \left(\psi_{k,\ell}^{i,\nu}\right)_{k,\ell} = \left(\varphi^{(i)}_{g_\nu(k),g_\nu(\ell)}\right)_{k,\ell}$ is essentially the matrix P_i^ν in the cone-coordinates $\boldsymbol{\lambda}$ on $\mathfrak{C}_\nu$. From (5) it follows that $\mathbf{x}^T P_i^\nu \mathbf{x} > 0$ is equivalent to Ψ_i^ν being strictly copositive; recall that a matrix $M \in \mathbb{R}^{n\times n}$ is said to be copositive if $\mathbf{x}^T M \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}_+^n$ and strictly copositive if $\mathbf{x}^T M \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$.

In [5] it is shown that the V_i are well-defined and continuous when they are defined in this way, i.e. if $\mathbf{x} \in \mathfrak{C}_\nu \cap \mathfrak{C}_\mu$, $\nu, \mu \in I$, the value $V_i(\mathbf{x})$ in (5) does not depend on whether one uses $\mathfrak{C}_\nu$ or $\mathfrak{C}_\mu$.

Now let us turn to how to express the constraints (2) in Theorem 1 for our piecewise quadratic Lyapunov functions. It is not difficult to see that if $V_i(\mathbf{x}) = \mathbf{x}^T P_i^\nu \mathbf{x}$ on $\mathfrak{C}_\nu$, then the conditions in (2) are equivalent to: for every $\nu \in I$ we have for every $\mathbf{x} \in \mathfrak{C}_\nu \setminus \{\mathbf{0}\}$ and every $i, j \in \mathcal{P}$ that

$$\mathbf{x}^T P_i^\nu \mathbf{x} > 0, \tag{6a}$$

$$\mathbf{x}^T \left(A_i^T P_i^\nu + P_i^\nu A_i + \alpha P_i^\nu\right) \mathbf{x} \leq 0, \tag{6b}$$

$$\mathbf{x}^T \left(P_i^\nu - \mu P_j^\nu\right) \mathbf{x} \leq 0. \tag{6c}$$

For (6b) just note that for every $\mathbf{x} \in \mathbb{R}^n$ there must be at least one $\mathfrak{C}_\nu$ such that $\mathbf{x} + [0,h] A_i \mathbf{x} \in \mathfrak{C}_\nu$ for a small enough $h > 0$ and then

$$\begin{aligned} D^+ V_i(\mathbf{x}, A_i\mathbf{x}) &= \limsup_{h\to 0+} \frac{V_i(\mathbf{x} + hA_i\mathbf{x}) - V_i(\mathbf{x})}{h} \\ &= \limsup_{h\to 0+} \frac{(\mathbf{x} + hA_i\mathbf{x})^T P_i^\nu (\mathbf{x} + hA_i\mathbf{x}) - \mathbf{x}^T P_i^\nu \mathbf{x}}{h} \\ &= \mathbf{x}^T \left(A_i^T P_i^\nu + P_i^\nu A_i\right) \mathbf{x}. \end{aligned}$$

Note that if $\mu > 1$ we do not have to explicitly enforce the constraints (6a) if, for every $\nu \in I$, we take care that for at least one $i \in \mathcal{P}$ we have $\mathbf{x}^T P_i \mathbf{x} \neq 0$ for all $\mathbf{x} \in \mathfrak{C}_\nu \setminus \{\mathbf{0}\}$. That is, $\mathbf{x}^T P_j^\nu \mathbf{x} > 0$ for all $j \in \mathcal{P}$ and all $\mathbf{x} \in \mathfrak{C}_\nu \setminus \{\mathbf{0}\}$ is then implicitly enforced by the conditions (6c). To see this, assume for a contradiction that $\mathbf{x}^T P_i \mathbf{x} < 0$ for some $\mathbf{x} \in \mathfrak{C}_\nu \setminus \{\mathbf{0}\}$. Then, by (6c) we have

$$\mathbf{x}^T (\mu P_j) \mathbf{x} \geq \mathbf{x}^T P_i \mathbf{x} > \mu \cdot \mathbf{x}^T P_i \mathbf{x} = \mathbf{x}^T (\mu P_i) \mathbf{x} \geq \mathbf{x}^T P_j \mathbf{x},$$

which is impossible because $0 > \mu \cdot \mathbf{x}^T P_i \mathbf{x} \geq \mathbf{x}^T P_j \mathbf{x}$ and $\mu > 1$. Hence, $\mathbf{x}^T P_i \mathbf{x} > 0$ and $\mathbf{x}^T P_j \mathbf{x} \geq \mu^{-1} \mathbf{x}^T P_i \mathbf{x} > 0$ for all $\mathbf{x} \in \mathfrak{C}_\nu \setminus \{\mathbf{0}\}$.

To express the constraints (6b) and (6c) as linear constraints in our variables for our LP problem it is advantageous to define the matrices

$$\begin{aligned} Q_i^\nu &:= X_\nu^T \left(A_i^T P_i^\nu + P_i^\nu A_i + \alpha P_i^\nu\right) X_\nu \\ \text{and} \qquad R_{i,j}^\nu &:= X_\nu^T \left(P_i^\nu - \mu P_j^\nu\right) X_\nu \end{aligned}$$

for all $\nu \in I$ and all $i, j \in \mathcal{P}$. Note that in the cone coordinates $\boldsymbol{\lambda} = X_\nu^{-1}\mathbf{x}$ we have

$$\begin{aligned} \boldsymbol{\lambda}^T Q_i^\nu \boldsymbol{\lambda} &= \mathbf{x}^T \left(A_i^T P_i^\nu + P_i^\nu A_i + \alpha P_i^\nu\right) \mathbf{x} \\ \text{and} \quad \boldsymbol{\lambda}^T R_{i,j}^\nu \boldsymbol{\lambda} &= \mathbf{x}^T \left(P_i^\nu - \mu P_j^\nu\right) \mathbf{x} \end{aligned}$$

so (6b) is equivalent to $-Q_i^\nu$ being copositive and (6c) is equivalent to $-R_{i,j}^\nu$ begin copositive.

For asserting the copositivity of the matrices through linear constraints we use the following theorem proved in [20].

Theorem 2: Let $B \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $C \in \mathbb{R}^{n\times n}$ be a symmetric matrix such that $c_{k,k} = b_{k,k}$, $c_{k,\ell} \geq b_{k,\ell}$, $c_{k,\ell} \geq 0$, for all $k, \ell = 1, 2, \ldots, n$, $k \neq \ell$. Then $-B$ is copositive if $\sum_{\ell=1}^n c_{k,\ell} \leq 0$ for all $k = 1, 2, \ldots, n$. If the inequality is strict for all k, then $-B$ is strictly copositive. $\square$

We close this section by summing up how we compute CPQ Lyapunov functions for the system (1) as in Theorem 1 and writing out the components of the relevant matrices Ψ_i^ν, Q_i^ν, and $R_{i,j}^\nu$ in the variables $\varphi_{k,\ell}^{(i)}$.

For the system (1) we use LP and Theorem 2 to obtain copositive $n\times n$-matrices Q_i^ν and $R_{i,j}^\nu$, $\nu = 1, 2, \ldots, N$ and $i, j \in \mathcal{P}$, where

$$Q_i^\nu = \left(q_{k,\ell}^{i,\nu}\right)_{k,\ell} \quad \text{and} \quad R_{i,j}^\nu = \left(r_{k,\ell}^{i,j,\nu}\right)_{k,\ell},$$

have, with $\hat{A}_i^\nu = \left(\hat{a}_{k,\ell}^{i,\nu}\right)_{k,\ell} := X_\nu^{-1} A_i X_\nu \in \mathbb{R}^{n\times n}$, the components

$$\begin{aligned} q_{k,\ell}^{i,\nu} = \Big(&\sum_{r=1}^n \hat{a}_{r,k}^{\nu,i} \varphi^{(i)}_{g_\nu(r),g_\nu(\ell)} + \hat{a}_{r,\ell}^{\nu,i} \varphi^{(i)}_{g_\nu(k),g_\nu(r)}\Big) \\ &+ \alpha \varphi^i_{g_\nu(k),g_\nu(\ell)} \end{aligned}$$

and

$$r_{k,\ell}^{i,j,\nu} = \varphi^{(i)}_{g_\nu(k),g_\nu(\ell)} - \mu \varphi^{(j)}_{g_\nu(k),g_\nu(\ell)}.$$

Additionally, we enforce $\Psi_i^\nu = \left(\varphi^{(i)}_{g_\nu(k),g_\nu(\ell)}\right)_{k,\ell}$ to be strictly copositive for $i = 1$, again using LP and Theorem 2. If we obtain a feasible solution to our LP problem for some given parameters $\alpha > 0$ and $\mu \geq 1$, i.e. are successful in generating the copositive matrices Q_i^ν and $R_{i,j}^\nu$, then the Lyapunov functions V_i for the subsystems $\dot{\mathbf{x}} = A_i\mathbf{x}$ are defined using the matrices $P_i^\nu = (X_\nu^{-1})^T \Psi_i^\nu X_\nu^{-1}$, and the origin is GES for every switching signal with MADT $\tau > \ln(\mu)/\alpha$. In particular, if $\mu = 1$, we have $V_1 = V_2 = \cdots = V_{N_{subs}} =: V$, the function V is a CLF for all the subsystems $\dot{\mathbf{x}} = A_i\mathbf{x}$, $i \in \mathcal{P}$, and the origin is GES under arbitrary switching.

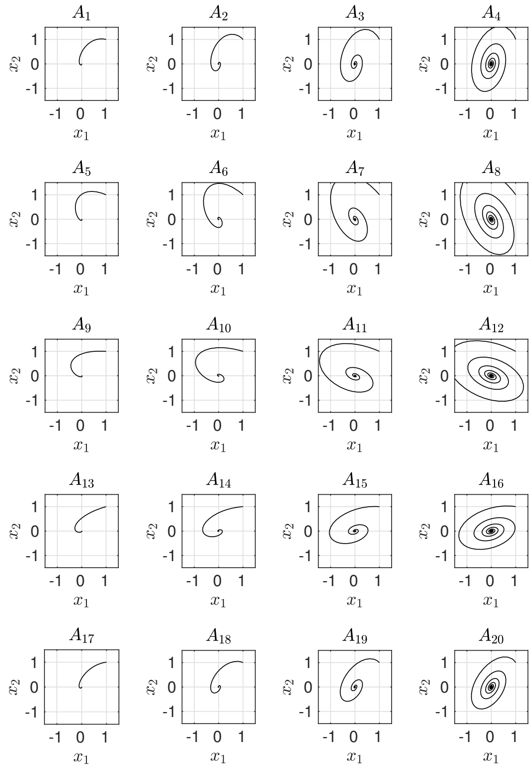

Fig. 2: Trajectories of the subsystems considered starting at $(1,1)$. The system matrices are defined in [2].

III. Numerical Experiments

In this section we compare our method to compute CPQ Lyapunov functions as in Theorem 1 using LP to the method in [4], where LP is used to compute CPA Lyapunov functions. The solver we use in all cases is Gurobi [12]. We consider the switched linear system (1) with $N_{\text{subs}} \in \{2,3,4\}$ subsystems in our comparison. The subsystems are chosen from the 20 planar systems defined in [2], where the formulas for the matrices A_i are given; their trajectories can be seen in Fig. 2. We use all possible ${}_{20}\mathrm{C}_2 = 190$ combinations of two subsystems, all ${}_{20}\mathrm{C}_3 = 1,140$ combinations of three subsystems, and all ${}_{20}\mathrm{C}_4 = 4,845$ combinations of four subsystems; altogether $6,175$ switched systems.

We want to get a picture of what is happening for both high and low resolutions of the triangulation. We performed the computations using the triangulations $\mathcal{T}_K^{\mathbf{F}}$ with the scaling factors $K \in \{1,2,\dots,10\}$; in the plane $\mathcal{T}_K^{\mathbf{F}}$ consists of $8K$ triangles. For each switched system we first try to find a CLF for the subsystems, i.e. we use $\mu = 1$ and a small $\alpha > 0$, see also [3], [20] for such methods for CPA- and CPQ CLFs, respectively. If a CLF can be computed for the subsystems, then the origin is GES for the switched system under arbitrary switching, i.e. with a MADT equal to zero.

If we are not successful in computing a CLF for the subsystems, then we use the algorithm from [4] to find a MADT $\tau > 0$ that asserts GES of the origin for the switched system. We are interested in computing as small MADT $\tau > 0$ as possible, and the algorithm searches the (α,μ)-plane for values such that we obtain a feasible solution to the LP problem with $\tau = \ln(\mu)/\alpha$ as small as possible. The feasibility problem can either be the LP problem described in the last section to obtain CPQ Lyapunov functions or the LP problem from [4] to compute CPA Lyapunov functions. The algorithm walks along and jumps between level sets of $\tau(\alpha,\mu) := \ln(\mu)/\alpha$, see (3), in order to efficiently search for the MADT needed for stability. It is defined as follows:

Algorithm 3: We fix $N, M \in \mathbb{N}$, $N \geq 2$, and $a, b, \overline{\alpha} > 0$ and distribute the values $a = \tau_N < \tau_{N-1} < \ldots < \tau_2 < \tau_1 = b$ and $0 < \alpha_1 < \ldots < \alpha_M < \overline{\alpha}$ uniformly, i.e.

$$\tau_j = (a-b)\frac{j-1}{N-1} + b \quad \text{and} \quad \alpha_i = \overline{\alpha}\frac{i}{M+1}$$

for $j = 1,2,\dots,N$ and $i = 1,2,\dots,M$.

Set $i = 1$ and $j = 1$. Then execute:

1) Solve the feasibility problem with $\alpha = \alpha_i$ and $\mu = \exp(\tau_j \alpha_i)$.
2)
   - If there is a feasible solution to the problem, then increase j by one, i.e. $j \leftarrow j+1$, and go back to step (1). That is, we decrease τ from τ_j to τ_{j+1} and check if this problem, with a lower MADT and a smaller $\mu = \exp(\tau_{j+1}\alpha_i)$, also has a feasible solution.
   - If there does not exists a feasible solution to the problem, then increase i by one, i.e. $i \leftarrow i+1$, and go back to step (1). That is, we increase α from α_i to α_{i+1} and set $\mu = \exp(\tau_j \alpha_{i+1})$ and check if this problem, with the same MADT $\tau = \tau_j$, but with larger parameters α and μ, has a feasible solution.

This algorithm is discussed in detail in [4]. A strict upper bound $\overline{\alpha}$, i.e. the optimal interval $[0,\overline{\alpha}]$ on the α-axis, can be computed a priori depending on the subsystems and the type of Lyapunov functions and the search for optimal pairs (α,μ), such that $\tau = \ln(\mu)/\alpha$ is as low as possible, is much more efficient than an exhaustive search and with almost as good results.

In our experiments, the total number of variables in the LP problem to compute CPQ Lyapunov functions, when the switched system has two, three or four subsystems, is roughly 3.8, 4.8 and 5.8 times the number of variables in the LP problem to compute CPA Lyapunov functions, respectively. The number of constraints in the LP problem to compute CPQ Lyapunov functions is in all cases roughly 2.6 times the number of constraints in the LP problem to compute CPA Lyapunov functions.

A. Results

Since our method first attempts to compute a CLF, we get a comparison of the efficiency in computing CPQ- and CPA CLFs for a given resolution K of the triangulation. The results are summarized in Table I. The LP problem for a CPQ CLF always has a feasible solution if the LP problem for a CPA CLF has a feasible solution and often, for a given resolution parameter K, one can generate a CPQ CLF but not a CPA CLF. Note that for large enough K the LP for a CPA CLF will have a feasible solution and

TABLE I: The number of switched systems, where the LP for CPQ CLF ($\mu = 1$) had a feasible solution for the same, or lower, triangulation scaling factor K than the LP for CPA CLF. The results are split by the number of subsystems N_{subs} along with the overall count on the right. Note that in some cases (summarized in the bottom line) we found no CPA CLF with $K \leq 10$ although we could compute a CPQ CLF.

N_{subs}	2	3	4	SUM
Same K	18	13	2	33
Lower K	109	389	741	1239
no CPA CLF	15	120	397	672

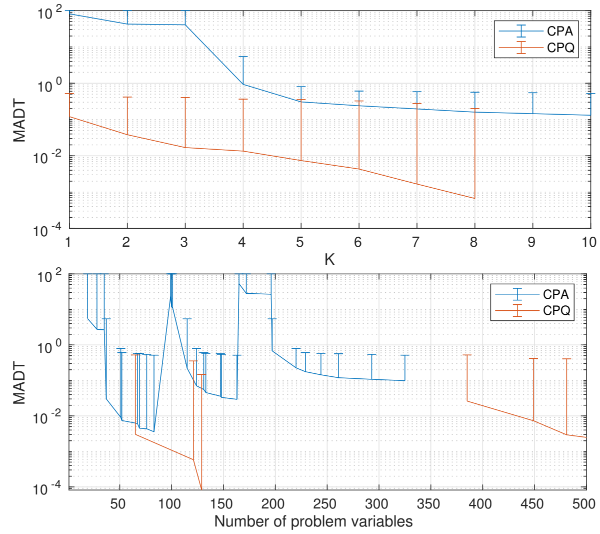

Fig. 3: Mean value of the MADT needed to assert GES of the origin as a function of the triangulation scaling factor K (top) and the number of problem variables in the LP problem (bottom), but only for the systems where we were able to generate a CPQ CLF. Comparison between the CPQ Lyapunov functions and CPA Lyapunov functions approaches. The error bars show the maximum values in each case. Note the logarithmic scale of the MADT-axis. The minimum values were all equal to zero, i.e. there was always a switched system where we could assert the GES of the origin under arbitrary switching.

also parameterize a CLF, see [13] for a proof. In Fig. 3 we plot the computed mean MADT, needed to assert GES of the origin for the switched system, as a function of the scaling parameter K. We do this using all the switched systems, for which we were able to compute a CPQ CLF with $K \leq 10$, and we compare the CPQ Lyapunov functions approach and the CPA Lyapunov functions approach. For Algorithm 3 we computed the optimal $\overline{\alpha}$ as described in [4] and set $[a, b] = [\tau_{\min}, \tau_{\max}] = [0, 25]$, except for the CPA Lyapunov functions approach with low K, where we set $b = \tau_{\max} = 100$ and used $\tau = \tau_{\max} = 100$ for the statistics when we did not get a feasible solution. Additionally, we plot the maximum values for the needed MADT using error bars; the minimum value is always zero. The results are clear: the CPQ Lyapunov function approach clearly outperforms the CPA Lyapunov function approach in computing a CLF, when one exist, and even if a CLF cannot be computed for a given K, then the CPQ Lyapunov function approach usually delivers a much lower MADT needed to assert GES.

Now, let us turn to the switched systems, for which we were not able to compute a CLF. In most or all of these cases we may assume, that a MADT restriction $\tau > 0$ is needed on the switching for GES of the origin for the switched system. In Fig. 4 we plot the computed mean MADT needed to assert GES of the origin as a function of the scaling parameter K. We do this using all the switched systems, for which we were *not* able to compute a CPQ CLF with $K \leq 10$, and we compare the CPQ Lyapunov functions approach and the CPA Lyapunov functions approach. The parameters for Algorithm 3 were set as above. For low K the CPQ Lyapunov functions approach again clearly outperforms the CPA Lyapunov functions approach. Notably, the computed MADT needed to assert GES does improve very little for larger K in the CPQ Lyapunov functions approach, which means that it is sufficient to use small K to get good results. For small K the CPA Lyapunov functions approach delivers much higher MADT needed to assert GES of the origin for the switched system than the CPQ Lyapunov functions approach. The results improve with larger K and the message is: the CPA Lyapunov functions approach needs more refined triangulations to deliver acceptable results.

To analyse this further, we compare in Fig. 5 how often the CPQ Lyapunov function approach delivered a lower MADT needed to assert GES than the CPA Lyapunov function approach. We performed this analysis separately for switched system with two-, three- and four subsystems and we considered all the switched system, both those for which we were able to compute a CLF and those where we were not able to compute a CLF. The results are again very interesting. For low K, the advantage of the CPQ Lyapunov approach is larger for more complicated switched systems, i.e. with more subsystems.

IV. Conclusions

We presented an algorithm to parameterize continuous and piecewise quadratic (CPQ) Lyapunov functions for linear switched systems on a triangulation of the state-space. The algorithm generates a linear programming (LP) problem, of which every feasible solution parameterizes CPQ Lyapunov functions for the subsystems of the switched system that fulfil a compatibility condition such that one obtains a minimum average-dwell time (MADT) $\tau \geq 0$ sufficient to assert that the origin is globally exponentially stable (GES) for every switching fulfilling the MADT condition. We compared our novel algorithm in numerical experiments to an algorithm in the literature that uses LP to parameterize continuous and piecewise affine (CPA) Lyapunov functions. Our new algorithm compares very favourably, in particular for triangulations with few triangles and for switched systems with several subsystems. Further, our numerical experiments showed that CPQ Lyapunov functions are substantially su-

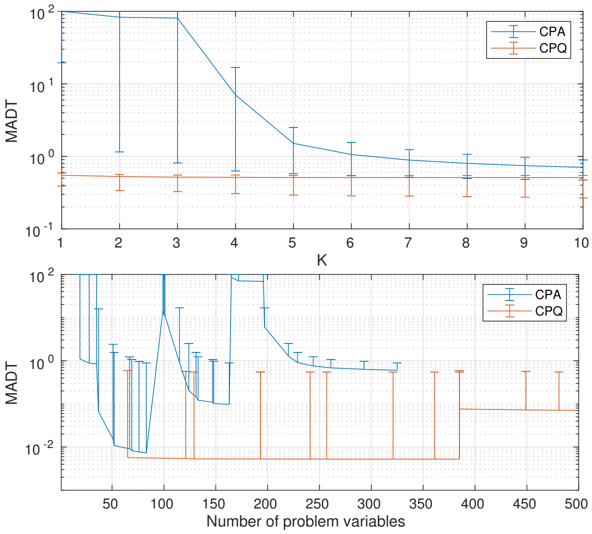

Fig. 4: Mean value of the MADT needed to assert GES of the origin as a function of the triangulation scaling factor K (top) and the number of variables in the LP problem (bottom), but now for the systems where we were *not* able to generate a CPQ CLF. Comparison between the CPQ Lyapunov functions approach and CPA Lyapunov functions approach. The error bars show the maximum- and minimum values in each case. Note the logarithmic scale of the MADT-axis.

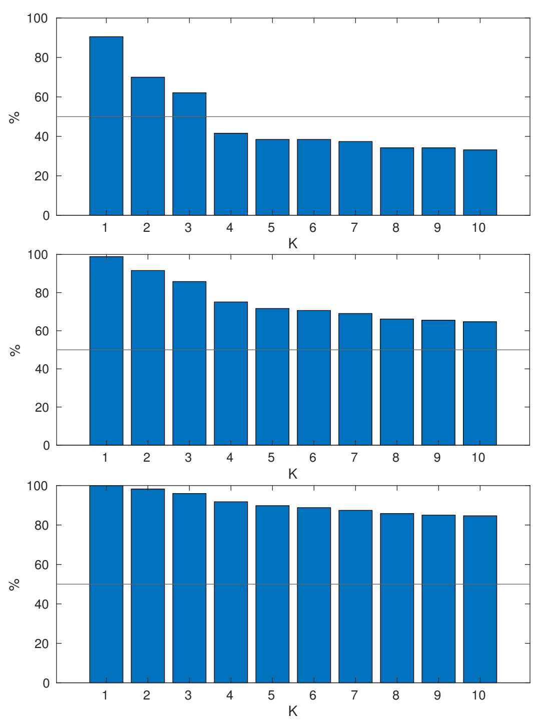

Fig. 5: Bar plot that shows the portion of systems where the CPQ Lyapunov functions approach delivered lower MADT needed to assert GES of the origin than the CPA Lyapunov functions approach for each of the triangulation scaling factors $K = 1, 2, \ldots, 10$. The switched systems have two (top), three (middle), and four (bottom) subsystems.

perior for computing common Lyapunov functions (CLFs) for the subsystems, asserting the GES of the origin under arbitrary switching, when they exist.

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