# Computation of Lyapunov functions using Zubov's equation and meshless collocation

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***Abstract*— A Lyapunov function can be used to find the domain of attraction of an equilibrium through its sublevel sets. In this paper, we propose a new method to compute a Lyapunov function by approximating the solution of Zubov's equation using meshless collocation. The advantage of approximating the solution of Zubov's equation compared to other equations characterising a Lyapunov function, is that for this particular Lyapunov function the domain of attraction is given by its sublevel set of level one, and thus we can obtain better approximations of the domain of attraction.**

## I. Introduction

We consider the system

$$\dot{x} = f(x), \quad f \in C^s(\mathbb{R}^d, \mathbb{R}^d),\ s \geq 1, \tag{1}$$

where $f(0) = 0$. The equilibrium at the origin is said to be exponentially stable, if for some $a > 0$ there exist constants $c, C > 0$ such that $\|\phi(t,x)\| \leq Ce^{-ct}\|x\|$ for all $\|x\| \leq a$ and all $t \geq 0$; here $\phi(t,x)$ denotes the solution to (1) at time $t$ with initial value $x$ at time 0, i.e. $\phi(0,x) = x$. It is well known that the origin is exponentially stable for (1), if and only if the Jacobian $Df(0) \in \mathbb{R}^{d\times d}$ of $f$ at the equilibrium at the origin is Hurwitz, i.e. the real parts of all eigenvalues of $Df(0)$ are negative; see e.g. [11, Corollary 4.3].

The domain of attraction for the equilibrium at the origin is defined as the set

$$A := \{x \in \mathbb{R}^d \colon \lim_{t\to\infty} \phi(t,x) = 0\},$$

and is a direct measure of the robustness of the stability of the equilibrium. We are interested in computing a good inner approximation of the set $A$ for a given system (1) with an exponentially stable equilibrium.

Lyapunov functions are functions such that the orbital derivative, i.e. the derivative along solutions of (1), is negative apart from the equilibrium and which have a minimum at the origin. In formulas, a Lyapunov function is a function $V \colon \mathbb{R}^d \to \mathbb{R}$ such that $V(0) = 0$, as well as $V(x) > 0$ and $\dot{V}(x) := \nabla V(x)f(x) < 0$ for all $x \in U \setminus \{0\}$, where $U$ is a neighborhood of 0. Then compact sublevel sets of $V$ in $U$ are subsets of the domain of attraction of the origin. For the purpose of finding an inner approximation of the domain of attraction, we aim to find a Lyapunov function such that its sublevel set approximates the domain of attraction as closely as possible.

Several methods have been proposed to compute Lyapunov functions, for a survey see [8], and to use them to estimate the domain of attraction, see e.g. [16], [15], [4], [5], [10]. One of these methods considers a Lyapunov function satisfying the following equation for its orbital derivative

$$\nabla V(x)f(x) = -\|x\|^2$$

and uses meshfree collocation to solve this (partial differential) equation for $V$, see [6], [9].

In this paper, we also use meshfree collocation, but we consider a different Lyapunov function, characterised by Zubov's equation. The advantage is that the solution of Zubov's equation determines the domain of attraction by the level set $\{x \in \mathbb{R}^d \colon V(x) < 1\}$ and is thus better suited to find good approximations of the domain of attraction.

Let us give an overview of the paper: In Section II we recall and state Zubov's theorem in the form most useful for our purpose. In Section III we present the method to approximately solve Zubov's equation using meshfree collocation, and thus to compute a Lyapunov function. Section IV contains error estimates, and in Section V the method is applied to two examples before we conclude in Section VI.

## II. Zubov's equation

Zubov showed in [20] that there exists a Lyapunov function, satisfying a certain PDE, which has the advantage of determining the boundary of the domain of attraction precisely, rather than finding subsets of the domain of attraction; see e.g. [2], [3], [1] for further discussions on Zubov's method. We use the following slightly altered version of his results, where we assume that the equilibrium is exponentially stable and thus can choose the first factor on the right-hand side in (2) as $\|x\|^2$. Moreover, we make the smoothness of $V$ explicit, which is needed for the numerical method in the next section.

*Theorem 1:* Consider the system (1) and assume that the origin is an exponentially stable equilibrium. Then an open neighbourhood $A \subset \mathbb{R}^d$ of the origin is the origin's domain of attraction, if and only if there exists a function $V \in C^s(A, \mathbb{R})$ such that

1) $V(0) = 0$,
2) $0 < V(x) < 1$ for all $x \in A \setminus \{0\}$,
3) $V(x_n) \to 1$ when $x_n \in A$ is a sequence with $x_n \to \partial A$ or $\|x_n\| \to \infty$,
4) $V$ is the unique solution with $V(0) = 0$ to

$$\nabla V(x)f(x) = -\|x\|^2(1 - V(x))\sqrt{1 + \|f(x)\|^2}. \tag{2}$$

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*Proof:* The proof follows many of the ideas in the proofs of Theorems 19 and 22 and Corollary 3 in [20]. Let $\phi\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ be the solution to the system

$$\dot{x} = \frac{f(x)}{\sqrt{1+\|f(x)\|^2}} =: g(x), \tag{3}$$

where clearly $\|g(x)\| \le 1$. Note that the origin is exponentially stable for (3), if and only if it is exponentially stable for $\dot{x} = f(x)$, because they have the same Jacobian $Df(0) = Dg(0)$ at the origin. Further, the systems are topologically equivalent and have the same trajectories, see e.g. [14, Ch. 3.1], and therefore the same domain of attraction for the equilibrium at the origin. The advantage of using system (3), instead of system (1), is that for any initial value $x \in A$ the solution to (3) exists for all $t \in \mathbb{R}$ and this is the reason we put the factor $\sqrt{1+\|f(x)\|^2}$ in (2), which translates to $\nabla V(x)g(x) = -\|x\|^2(1-V(x))$ for the system (3).

To show the *only if* part we explicitly construct a solution to (2) and show that it fulfills all the conditions stated. As the equilibrium at the origin is exponentially stable, we can define for every $x \in A$ the function $V$ using the formula

$$V(x) = 1 - \exp\left(-\int_0^\infty \|\phi(\tau,x)\|^2 d\tau\right).$$

This function obviously fulfills the conditions 1) and 2). To show that it fulfills the condition 3) first consider a sequence $x_n \in A$ such that $\|x_n\| \to \infty$ as $n \to \infty$. We have

$$\begin{aligned}\|\phi(t,x)\| &= \left\|x + \int_0^t \dot{\phi}(\tau,x)d\tau\right\| \\ &\ge \|x\| - \int_0^t \|g(\phi(\tau,x)\|d\tau \ge \|x\| - \int_0^t 1 d\tau \ge \|x\| - t\end{aligned}$$

and thus

$$\begin{aligned}\int_0^\infty \|\phi(\tau,x)\|^2 d\tau &\ge \int_0^{\|x\|} (\|x\|-\tau)^2 d\tau \ge \frac{1}{3}\|x\|^3 \\ V(x_n) &\ge 1 - \exp\left(-\frac{1}{3}\|x_n\|^3\right) \to 1\end{aligned}$$

as $n \to \infty$.

Now assume that $\partial A \neq \emptyset$ and assume $x_n \in A$ is a bounded sequence such that $x_n \to \partial A$, i.e. $\operatorname{dist}(x_n, \partial A) := \inf_{y\in\partial A}\|x_n - y\| \to 0$, as $n \to \infty$. Assume $V(x_n) \not\to 1$ as $n \to \infty$ for a contradiction. Then there is a $\varepsilon > 0$ and a subsequence $x_n^*$ of the $x_n$ such that $V(x_n^*) < 1-\varepsilon$ for all $n \in \mathbb{N}$ and because the sequence $x_n^*$ is bounded it has a convergent subsequence $y_n$ such that $y_n \to y \in \partial A$ as $n \to \infty$. With $D := \operatorname{dist}(0, \mathbb{R}^n \setminus A) > 0$ and since $\mathbb{R}^n \setminus A$ is forward invariant, we have $\|\phi(t,y)\| \ge D$ for all $t \ge 0$. Let $L > 0$ be a Lipschitz constant for the continuously differentiable function $g$ on the closed ball with radius $r := \|y\| + D/2$ and centered at the origin. Choose $T > 0$ so large that $\exp(-TD^2/4) \le \varepsilon/2$. Then for $\|y-x\| \le \delta := e^{-LT}D/2$ we have $\|\phi(t,y) - \phi(t,x)\| \le e^{Lt}\|y-x\| \le D/2$ for all $t \in [0,T]$. Then for $y_n$ such that $\|y_n - y\| \le \delta$ we have

$$\begin{aligned}\int_0^\infty \|\phi(\tau,y_n)\|^2 d\tau &\ge \int_0^T (\|y\| - D/2)^2 d\tau \ge TD^2/4 \\ V(y_n) &\ge 1 - \exp\left(-TD^2/4\right) \ge 1 - \frac{\varepsilon}{2},\end{aligned}$$

a contradiction. We have shown that 3) must hold true.

To show that $V$ fulfills the equation 4) on $A$ consider that because $\phi(\tau,\phi(t,x)) = \phi(\tau+t,x)$ we have

$$\begin{aligned}1 - V(x) &= \exp\left(-\int_0^\infty \|\phi(\tau,x)\|^2 d\tau\right) \\ &= \exp\left(-\int_0^t \|\phi(\tau,x)\|^2 d\tau - \int_t^\infty \|\phi(\tau,x)\|^2 d\tau\right) \\ &= \exp\left(-\int_0^t \|\phi(\tau,x)\|^2 d\tau\right)\exp\left(-\int_0^\infty \|\phi(\tau+t,x)\|^2 d\tau\right) \\ &= \exp\left(-\int_0^t \|\phi(\tau,x)\|^2 d\tau\right)\exp\left(-\int_0^\infty \|\phi(\tau,\phi(t,x))\|^2 d\tau\right) \\ &= \exp\left(-\int_0^t \|\phi(\tau,x)\|^2 d\tau\right)(1 - V(\phi(t,x)))\end{aligned}$$

and then, by isolating $V(\phi(t,x))$ and taking the derivative with respect to $t$, we get

$$\begin{aligned}\nabla V(x)\cdot\frac{f(x)}{\sqrt{1+\|f(x)\|^2}} &= \frac{d}{dt}V(\phi(t,x))\Big|_{t=0} \\ &= \frac{d}{dt}\left[\exp\left(\int_0^t \|\phi(\tau,x)\|^2 d\tau\right)(V(x)-1)+1\right]\Big|_{t=0} \\ &= \|x\|^2[V(x)-1],\end{aligned}$$

i.e.

$$\nabla V(x)f(x) = -\|x\|^2(1-V(x))\sqrt{1+\|f(x)\|^2}.$$

Since $\phi \in C^s(\mathbb{R}\times\mathbb{R}^d, \mathbb{R}^d)$, see e.g. [12, Ch. 6.2] or [17, III.§13], and one can differentiate under the integral, we can conclude that $V \in C^s(A,\mathbb{R})$, following the arguments in the proof of [6, Theorem 2.46].

For the rest of the proof we will denote the domain of attraction by $A^*$. To show that $V$ is the unique solution to the PDE (2) with $V(0) = 0$ we show, for later use, that this holds true for any forward invariant subset of the domain of attraction $A \subset A^*$, i.e. $x \in A$ implies $\phi(t,x) \in A$ for all $t \ge 0$. Assume $W$ is a further solution on $A$ with $W(0) = 0$ and consider the function $U(t) = V(\phi(t,x)) - W(\phi(t,x))$. We have

$$\lim_{t\to\infty} U(t) = 0 \tag{4}$$

and, by (2), that $U$ satisfies the ODE $U'(t) = \|\phi(\tau,x)\|^2 U(t)$ with solution $U(t) = U(0)\exp(\int_0^t \|\phi(\tau,x)\|^2 d\tau)$. Then $|U(t)| \ge |U(0)|$ for all $t \ge 0$, and thus (4) implies $U(0) = 0$, i.e. $V(x) = W(x)$. Since the domain of attraction $A^*$ is forward invariant, we can choose $A = A^*$ and the uniqueness on $A^*$ follows.

To show the *if* part assume now that there is a function $V \in C^s(A,\mathbb{R})$ fulfilling 1)-4) and consider $V^{-1}([0,c]) \subset A$ for a $0 < c < 1$. Because of condition 3) the set $V^{-1}([0,c]) \subset A$ is closed in $\mathbb{R}^d$ and bounded. That

$V^{-1}([0,c])$ is a subset of the domain of attraction now follows, e.g., as in the proof of [11, Theorem 4.10] and since $A = \bigcup_{0<c<1} V^{-1}([0,c])$ by condition 3). It is also clear that $A$ is forward invariant, i.e. $x \in A$ implies $\phi(t,x) \in A$ for all $t \geq 0$. Hence, we can use the result above about the uniqueness on $A$.

To show that $A$ is indeed the origin's domain of attraction, we now assume that $A$ is a proper subset of the domain of attraction and lead this to a contradiction. Let $x \in A^* \setminus A$ and note that there is a $t_0 \geq 0$ such that $x_0 = \phi(t_0, x) \in \partial A \cap A^*$. Let $W$ be the function from the *only if* part of the proof (which was there denoted by $V$). $W$ is defined in the domain of attraction and satisfies $W(x_0) < 1$. By assumption $V(x_0^n) \to 1$ for every sequence $(x_0^n)_{n\in\mathbb{N}}$ in $A$ converging to $x_0 \in \partial A$ and we obtain the contradiction

$$1 > W(x_0) = \lim_{n\to\infty} W(x_0^n) = \lim_{n\to\infty} V(x_0^n) = 1$$

from the uniqueness on the forward invariant subset $A$ of the domain of attraction. ∎

We propose solving the equation (2) numerically using meshfree collocation.

## III. Solving Zubov's equation via meshfree collocation

We will use a linear combination of compactly supported radial basis functions (RBFs) to approximate a solution to Zubov's equation (2) and this linear combination will vanish outside of the joint support of the RBFs. Hence, we reformulate Theorem 1 and consider the function

$$W(x) := V(x) - 1,$$

for which we have:

1) $W(0) = -1$,
2) $-1 < W(x) < 0$ for all $x \in A \setminus \{0\}$,
3) $W(x_n) \to 0$ when $x_n \in A$ is a sequence with $x_n \to \partial A$ or $\|x_n\| \to \infty$,
4) $W$ is the unique solution with $W(0) = -1$ to
$$\nabla W(x) f(x) = \|x\|^2 W(x) \sqrt{1 + \|f(x)\|^2}.$$

In particular, $W$ can be extended continuously to $\mathbb{R}^d$ by setting $W(x) = 0$ for $x \in \mathbb{R}^d \setminus A$.

Denoting

$$h(x) = \|x\|^2 \sqrt{1 + \|f(x)\|^2}$$

we now seek to solve the PDE

$$\nabla W(x) f(x) - h(x) W(x) = 0, \quad \text{with } W(0) = -1 \tag{5}$$

for all $x \in A$. We use meshfree collocation with RBFs to approximate $W$ by $w$, similar to how Lyapunov functions are computed in [6], [9], [7].

Meshfree collocation is an established method to solve (generalized) interpolation problems such as partial differential equations, see e.g. [19]. Let $0 \in \Omega \subset A$ be a bounded domain with Lipschitz boundary. We choose a kernel given by a Wendland function $\psi_0(r) = \phi_{k,\ell}(cr)$, $c > 0$, $k \geq 2$ and $\ell = \lfloor d/2 \rfloor + k + 1$, see [18] for the definition of Wendland functions. The corresponding reproducing kernel Hilbert space $H$ with kernel $\Phi(x,y) = \psi_0(\|x - y\|)$, restricted to $(x,y) \in \Omega^2$ consists of functions $\Omega \to \mathbb{R}$ and has norm $\|\cdot\|_H := \sqrt{\langle \cdot, \cdot \rangle_H}$. In particular, for this choice of the kernel, it consists of the same functions as the Sobolev space $H^\tau(\Omega)$, $\tau = k + (d+1)/2$ and the spaces are norm equivalent, see e.g. [9]. We assume that $s \geq \tau$, i.e. the solution $W$ to (5) is in $H$.

We choose $N$ pairwise different collocation points $X = \{x_1, \ldots, x_N\} \subset \Omega \setminus \{0\}$, where we force the PDE (5) to be fulfilled. For this, we define the linear differential operator $L\colon H^\tau(\Omega) \to H^{\tau-1}(\Omega)$ by

$$LW(x) := \nabla W(x) f(x) - h(x) W(x).$$

In the next lemma we will determine the singular points of $L$, i.e. points $x \in \Omega$ for which $\delta_x \circ L = 0$.

*Lemma 1:* The only singular point of $L$ in $\Omega$ is the equilibrium at 0.

*Proof:* Let $x \in \Omega \setminus \{0\}$. Then the function $W \equiv 1$ satisfies $LW(x) = -h(x) \neq 0$. For $x = 0$, however, we have for any function $W$ that

$$LW(0) = \nabla W(0) f(0) - h(0) W(0) = 0$$

as $f(0) = 0$ and $h(0) = 0$. ∎

Moreover, we define the following elements of the dual of $H$, namely $L_n \in H^*$, $n = 0, 1, \ldots, N$, through

$$L_0 w = w(0)$$

and, for $n = 1, 2, \ldots, N$,

$$L_n w = (\delta_{x_n} \circ L) w = \nabla w(x_n) f(x_n) - h(x_n) w(x_n).$$

Note that the $L_n$ are in $H^*$ and are linearly independent with a similar argumentation as in [9, Proposition 3.3 and 3.8] since $\tau = k + (d+1)/2 > 1 + d/2$; note that the only singular point 0 of the differential operator $L$ is not in $X$.

After this set-up, we now consider the approximation by meshfree collocation. It is well known that the solution to the generalized interpolation problem

$$\begin{cases} \text{min} & \|w\|_H \\ \text{such that} & L_0(w) = -1, \\ & L_n(w) = 0 \text{ for } n = 1, 2, \ldots, N \end{cases}$$

is given by

$$w(x) = \sum_{n=0}^{N} \beta_n L_n^y \Phi(x,y), \tag{6}$$

where $L_n^y$ denotes the application of the operator with respect to the variable $y$, i.e. for a fixed $x \in \Omega$ the operator $L_n$ is applied to the function $\Phi(x, \cdot) \in H$, and $\Phi$ was defined above. The coefficients $\beta_n \in \mathbb{R}$ are the solution to the system of linear equations $A\beta = (-1, 0, 0, \ldots, 0)^T \in \mathbb{R}^{N+1}$, where the matrix $A = (a_{ij})_{i,j=0,1,\ldots,N}$ is given by

$$a_{ij} = L_i^x L_j^y \Phi(x,y).$$

Since the $L_n$ are linearly independent, the matrix $A$ is positive definite.

We now compute the coefficients $a_{ij}$ as well as an explicit expression for $w(x)$ and $\dot{w}(x) := \nabla w(x) f(x)$.

We define $\psi_i(r) := \frac{\psi'_{i-1}(r)}{r}$ for $r > 0$ and $\psi_i(0) = 0$, where $i = 1, 2$. These functions are continuous since $k \geq 2$, see [6, Proposition 3.11]. We have

$$L_0^y \Phi(x, y) = \Phi(x, 0) = \psi_0(\|x\|)$$

and for $n \geq 1$ we have

$$\begin{aligned} L_n^y \Phi(x, y) &= \nabla_y \Phi(x, y) \cdot f(y) - h(y)\Phi(x, y)\big|_{y=x_n} \\ &= \psi_1(\|x - y\|)(-f(y) \cdot (x - y))\big|_{y=x_n} \\ &\qquad - h(x_n)\psi_0(\|x - x_n\|) \\ &= -\psi_1(\|x - x_n\|)(x - x_n) \cdot f(x_n) \\ &\qquad - h(x_n)\psi_0(\|x - x_n\|). \end{aligned}$$

Hence, by formula (6), we have

$$\begin{aligned} w(x) =& \beta_0 \psi_0(\|x\|) \\ & - \sum_{n=1}^{N} \beta_n \Big[ \psi_1(\|x - x_n\|)(x - x_n) \cdot f(x_n) \\ & \qquad + h(x_n)\psi_0(\|x - x_n\|) \Big] \end{aligned}$$

and

$$\begin{aligned} \dot{w}(x) &= \beta_0 \psi_1(\|x\|) x \cdot f(x) \\ & - \sum_{n=1}^{N} \beta_n \Big[ \psi_1(\|x - x_n\|) f(x) \cdot f(x_n) \\ & + \psi_2(\|x - x_n\|)[(x - x_n) \cdot f(x)][(x - x_n) \cdot f(x_n)] \\ & + h(x_n)\psi_1(\|x - x_n\|)(x - x_n) \cdot f(x) \Big]. \end{aligned}$$

For the matrix elements we have for $i, j \in \{1, \ldots, N\}$

$$\begin{aligned} a_{00} &= L_0^x L_0^y \Phi(x, y) = \psi_0(0), \\ a_{0j} &= L_0^x L_j^y \Phi(x, y), \\ &= \psi_1(\|x_j\|) x_j \cdot f(x_j) - h(x_j)\psi_0(\|x_j\|), \\ a_{i0} &= L_i^x L_0^y \Phi(x, y), \\ &= \psi_1(\|x_i\|) x_i \cdot f(x_i) - h(x_i)\psi_0(\|x_i\|), \\ a_{ij} &= L_i^x L_j^y \Phi(x, y), \\ &= -\psi_2(\|x_i - x_j\|)[(x_i - x_j) \cdot f(x_i)][(x_i - x_j) \cdot f(x_j)] \\ &\quad - \psi_1(\|x_i - x_j\|) f(x_i) \cdot f(x_j) \\ &\quad - \psi_1(\|x_i - x_j\|)(x_i - x_j) \cdot f(x_i) h(x_j) \\ &\quad + \psi_1(\|x_i - x_j\|)(x_i - x_j) \cdot f(x_j) h(x_i) \\ &\quad + \psi_0(\|x_i - x_j\|) h(x_i) h(x_j). \end{aligned}$$

## IV. Error estimates

The error estimate derived in the next theorem will ensure that for any sublevel set of the approximation $w$ with level lower than 0, we have $\dot{w}(x) < 0$ for all $x$ in the sublevel set apart from possibly an arbitrary small ball around the equilibrium.

*Theorem 2:* Fix a Wendland function with $k \geq 2$ such that $k + (d+1)/2 = \tau \leq s$. Fix a small open ball $B_\iota$ centred at the equilibrium at the origin with radius $\iota > 0$ as well as a bounded domain $\Omega$ with Lipschitz boundary such that $\overline{B_\iota} \subset \Omega \subset A$, where $A$ is the domain of attraction of 0. Define

$$\Delta := \min_{x \in \overline{\Omega} \setminus B_\iota} h(x) > 0.$$

Choose points $X \subset \Omega \setminus \{0\}$ such that the fill distance

$$h_X := \max_{x \in \Omega} \min_{n=1,2,\ldots,N} \|x - x_n\|$$

is so small that $\frac{\varepsilon}{\Delta} < 1$, where $\varepsilon := \tilde{C} h_X^{k-1/2}$ and $\tilde{C}$ is a constant depending only on $\tau$, $\Omega$ and $W$ and is defined below.

Choose the constant $\delta$ such that

$$\frac{\varepsilon}{\Delta} < \delta < 1$$

and define the sublevel set $\tilde{K} = \{x \in A \colon w(x) \leq -\delta\}$ as well as $K = \tilde{K} \setminus B_\iota$.

Then we have for the approximation $w$ via meshfree collocation that

$$\dot{w}(x) < 0$$

for all $x \in K$.

*Proof:* We will use the error estimate from [9, Theorem 3.10] with $m = 1$ and $\tau = k + (d+1)/2$. Note that we have $\lfloor \tau \rfloor \geq k + d/2 > 1 + d/2$. We obtain

$$\begin{aligned} \|LW - Lw\|_{L_\infty(\Omega)} &\leq Ch_X^{\tau - 1 - d/2} \|W\|_{H^\tau(\Omega)} \\ &= Ch_X^{k-1/2} \|W\|_{H^\tau(\Omega)} \\ &= \varepsilon \end{aligned}$$

where $\tilde{C} := C\|W\|_{H^\tau(\Omega)}$ is a constant depending only on $\tau$, $\Omega$ and $W$.

Then we have for all $x \in K$ that

$$\begin{aligned} \dot{w}(x) &= \dot{w}(x) - h(x)w(x) + h(x)w(x) \\ &\quad - \underbrace{(\dot{W}(x) - h(x)W(x))}_{=0 \text{ by } (5)} \\ &\leq |Lw(x) - LW(x)| + h(x)w(x) \\ &\leq \varepsilon - \Delta\delta \\ &< 0 \end{aligned}$$

since $\frac{\varepsilon}{\Delta} < \delta$. This shows the theorem. ■

## V. Examples

We apply our proposed method to two planar systems, where we know the domain of attraction beforehand. Note that in both examples the orbital derivative is negative near the origin, even though this is not guaranteed by Theorem 2.

### A. Unit circle

As our first example we consider the system given by

$$\begin{cases} \dot{x} = -x(1 - x^2 - y^2) + y \\ \dot{y} = -y(1 - x^2 - y^2) - x \end{cases} \tag{7}$$

It has an exponentially stable equilibrium at the origin and the domain of attraction is given by $\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$.

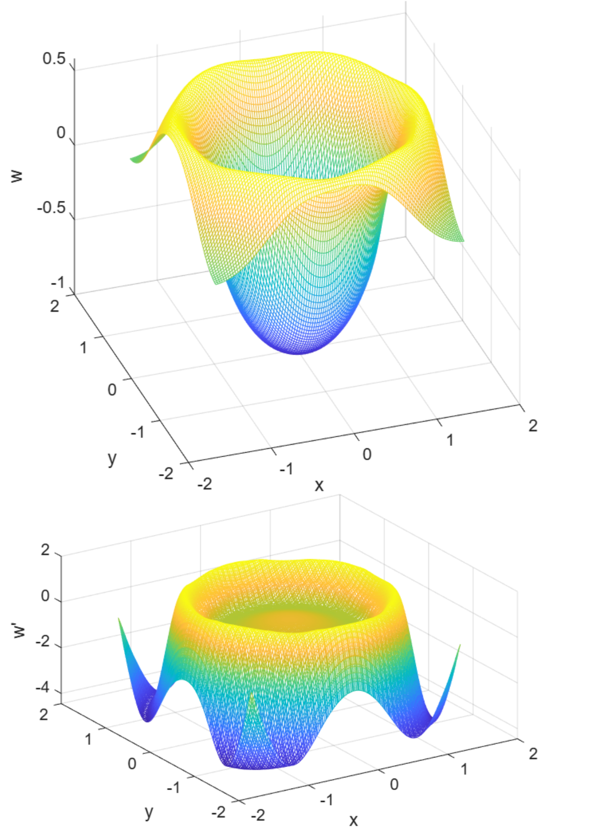

Fig. 1. Example (7). Top: the function $w(x, y)$, bottom: its orbital derivative $\dot{w}(x, y)$.

For the set $X$ we choose the 846 points of the form $\alpha(i + j/2, j\sqrt{3}/2)$ with $i, j \in \mathbb{Z}$ and $\alpha = 0.08$, which are contained in $\{(x, y) \in \mathbb{R}^2 \colon x^2+y^2 < 1.5\}\setminus\{(0, 0)\}$. We used the Wendland kernel given by $\psi_0(r) = \phi_{k,\ell}(cr)$ with $c = 0.9$ and $k = 4, \ell = 2$, i.e. $\phi_{4,2}(r) = (1 - r)^6_+(35r^2 + 18r + 3)$ where $(1 - r)^6_+ := (\max\{1 - r, 0\})^6$, see [18].

Figure 1 shows the function $w$ (top) as well as $\dot{w}$ (bottom). Figure 2 displays the set where $\dot{w} = 0$ as black dashed lines, as well as level sets of $w$ at levels $-0.5$, $-0.3$ and $-0.1$. The sublevel set $\{x \in \mathbb{R}^2 \colon w(x) \leq -0.1\}$ is a subset of the area where $\dot{w}(x) < 0$ and is thus a subset of the domain of attraction. It is a very good approximation of the actual domain of attraction (marked in blue), namely the unit disc.

### B. Van der Pol

The second example is the time-reversed van der Pol system given by

$$\left\{ \begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & x - (1 - x^2)y \end{array} \right. \tag{8}$$

It has an exponentially stable equilibrium at the origin and the boundary of its domain of attraction is given by an unstable periodic orbit.

For the set $X$ we choose the 1574 points of the form $\alpha(i + j/2, j\sqrt{3}/2)$ with $i, j \in \mathbb{Z}$ and $\alpha = 0.15$, which are contained in $\{(x, y) \in \mathbb{R}^2 \colon |x| < 2.5, |y| < 3\}\setminus\{(0, 0)\}$. The

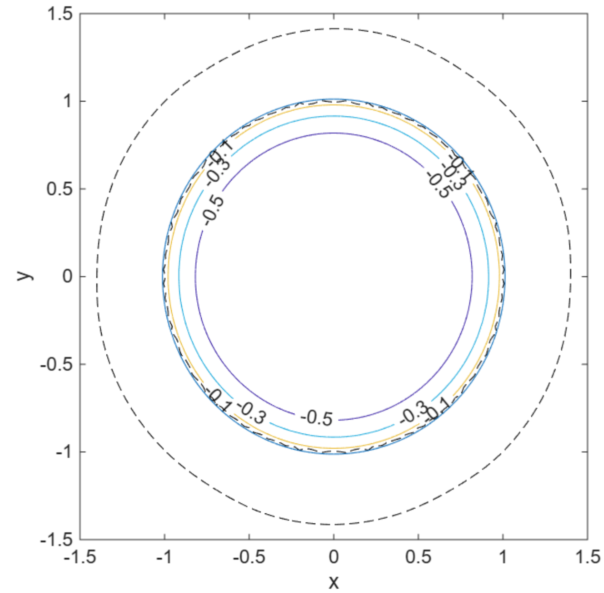

Fig. 2. Example (7). The figure shows level sets of $w$ at values $-0.5, -0.3, -0.1$ as well as the set where $\dot{w}$ is zero as black dashed line. The sublevel set $\{(x, y) \in \mathbb{R}^2 \colon w(x, y) \leq -0.1\}$ is fully contained in the set with negative orbital derivative and is thus a subset of the domain of attraction of the origin. The actual domain of attraction is bounded by the unit circle, marked in blue.

Wendland kernel is given by $\psi_0(r) = \phi_{k,\ell}(cr)$ with $c = 0.2$ and $k = 4, \ell = 2$.

Figure 3 shows the function $w$ (top) as well as $\dot{w}$ (bottom). Figure 4 displays the set where $\dot{w} = 0$ in black, as well as level sets of $w$ at levels $-0.5$, $-0.3$, $-0.1$ and $-0.03$. The sublevel set $\{x \in \mathbb{R}^2 \colon w(x) \leq -0.03\}$ is a subset of the area where $\dot{w}(x) < 0$ and is thus a subset of the domain of attraction. It is a reasonable approximation of the actual domain of attraction (marked in blue), which is bounded by the unstable periodic orbit.

## VI. Summary and Outlook

In this paper we have proposed a method to approximate the domain of attraction by computing a Lyapunov function, approximating the solution of Zubov's equation using meshfree collocation. Since the boundary of the domain of attraction is given by the level set of level 1 of the true solution of Zubov's equation, the approximation is able to approximate the domain of attraction very well. We have proven error estimates, demonstrating that using sufficiently dense collocation points, a suitably chosen sublevel set of the computed Lyapunov function lies in the area with negative orbital derivative.

The computational cost of our method is low as it is based on solving a system of linear equations of size $N + 1$, where $N$ is the number of collocation points. In comparison, many other methods to determine the domain of attraction such as SOS use LMI. However, the number of collocation points grows still exponentially with the dimension of the system. Refinement methods, which make use of the fact that the collocation points can be scattered, can improve this, see e.g. [13].

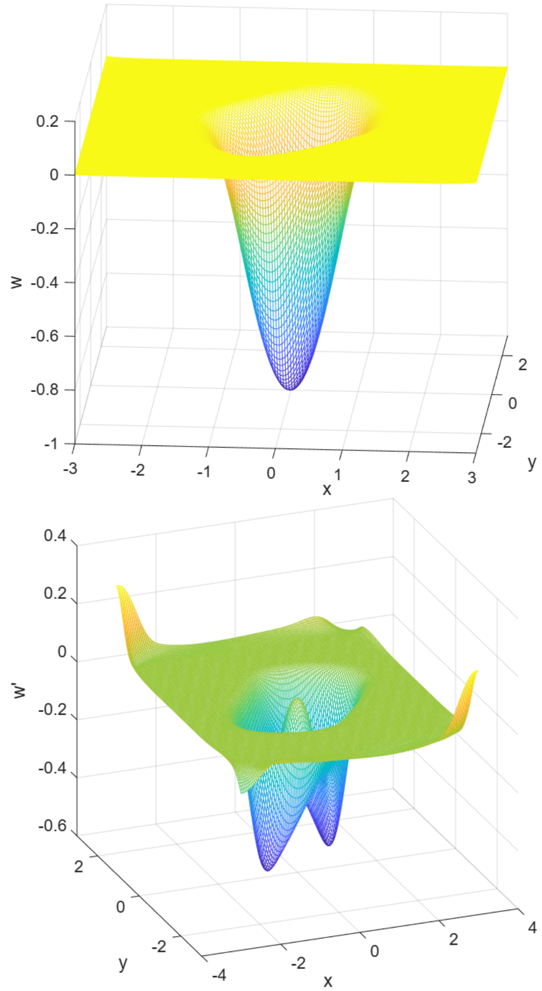


Fig. 3. Example (8). Top: the function $w(x, y)$, bottom: its orbital derivative $\dot{w}(x, y)$.

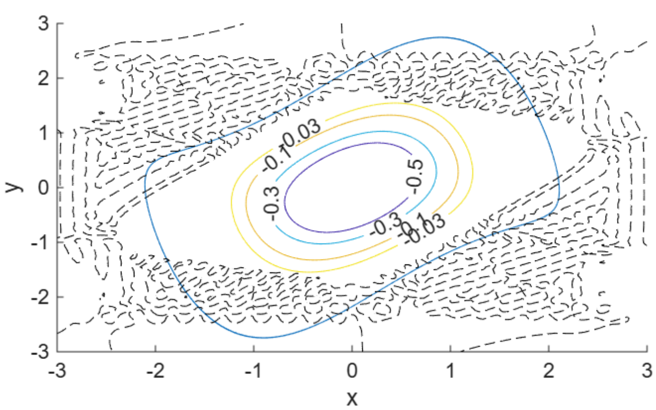


Fig. 4. Example (8). This figure shows level sets of $w$ at values $-0.5, -0.3, -0.1, -0.03$ as well as the set where $\dot{w}$ is zero as black dashed lines. The sublevel set $\{(x, y) \in \mathbb{R}^2 : w(x, y) \leq -0.03\}$ is fully contained in the set with negative orbital derivative and is thus a subset of the domain of attraction of the origin. The actual domain of attraction is bounded by an unstable periodic orbit, marked in blue.

For the error estimate we have assumed that the collocation points were placed inside the domain of attraction. In the examples, however, we have placed our collocation points in an area that is larger than the domain of attraction, as one might not know the domain of attraction a priori. In future work, it will be interesting to explore error estimates of a function that is extended beyond the domain of attraction; the difficulty is that such a function typically is only continuous at the boundary of the domain of attraction. Further, it is interesting that the approximate solution in our examples has a negative orbital derivative close to the equilibrium, although this is not guaranteed by our estimate in Theorem 2.

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