Stability Analysis of Hybrid Integrator-Gain Systems using Linear Programming ⋆

Juan Javier Palacios Roman[a,*], Sigurdur F. Hafstein[b], Sebastiaan J.A.M. van den Eijnden[a], Marcel F. Heertjes[a], W.P.M.H. Heemels[a,1]

[a] *Department of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, the Netherlands*
[b] *Faculty of Physical Sciences, University of Iceland, Reykjavik, Iceland*

Abstract

Analyzing the stability and performance of conewise linear systems, e.g., hybrid integrator-gain systems (HIGS), requires tools specifically designed for hybrid systems. For HIGS in particular, most results rely on constructing continuous piecewise quadratic (CPQ) Lyapunov functions by solving suitable sets of linear matrix inequalities. For the purpose of creating additional efficient methods for the analysis of HIGS-controlled systems and other conewise linear systems, a method for constructing continuous piecewise affine (CPA) Lyapunov functions with linear programming is investigated. We present the following main contributions: First, we extend the CPA method to the construction of input-to-state stable Lyapunov functions for conewise linear systems by using Farkas' lemma. Second, we show that the CPA method can also be used to calculate upper bounds on important performance metrics of conewise linear systems, specifically, the L_1-gain and

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*Corresponding author.

Email addresses: j.j.palacios.roman@tue.nl (Juan Javier Palacios Roman), shafstein@hi.is (Sigurdur F. Hafstein), s.j.a.m.v.d.eijnden@tue.nl (Sebastiaan J.A.M. van den Eijnden), m.f.heertjes@tue.nl (Marcel F. Heertjes), m.heemels@tue.nl (W.P.M.H. Heemels)

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H_1-norm. Third, a converse theorem is presented, which shows that, given a stable conewise linear system, the method presented in this paper always succeeds in finding a CPA Lyapunov function to the system provided that the CPA function is sufficiently refined. The CPA method is demonstrated in a numerical example for the stability and performance analysis of a HIGS-controlled system. In this example, the CPA method is compared to the CPQ method in terms of computational efficiency.

Keywords:
Hybrid integrator-gain system, Input-to-state stability, Linear programming, Lyapunov methods, L_1-gain, H_1-norm.

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1. Introduction

The demand for precision and throughput in high-tech motion systems is ever increasing. To facilitate these developments, controllers with progressively stricter performance requirements need to be designed. While linear time-invariant (LTI) controllers deliver high accuracy in a wide variety of applications, they suffer from inherent limitations in performance [1]. For this reason, the design and analysis of hybrid controllers has gained much traction, as they can potentially overcome these limitations for LTI plants.

Specific examples of hybrid controllers proposed for LTI plants are, among others, reset controllers [2, 3] and hybrid integrator-gain systems (HIGS) [4], the latter of which are of interest in this paper. Similar to an LTI integrator, these controllers induce a 20 dB/decade amplitude decay when evaluated with a frequency-domain approximation using a describing function. However, they differ in the amount of phase lag induced: Reset control and HIGS only induce $38.15°$ phase lag as opposed to $90°$ phase lag by an LTI integrator. This shows a potential benefit of these hybrid controllers over LTI integrators that could be used to achieve higher performance. Notably, reset control uses discontinuous control actions, which complicates stability analysis and could pose a performance problem for systems with higher-order dynamics. HIGS alleviates these problems as it provides a continuous control input by changing resets to projections. In previous work on HIGS-controlled systems, they have been shown to be well-posed [5, 6] and to indeed improve performance in high-precision motion systems, where previously LTI integrators were used [7, 8].

However, the tools for the stability and performance analysis of hybrid systems, like HIGS-controlled LTI systems, are much more limited than for LTI controllers. This is because a HIGS-controlled LTI system, i.e., an LTI plant in a feedback loop with HIGS, is described by a discontinuous conewise linear system [9], where some of the subdynamics are only active on a lower dimensional subset of the state space. Using the superposition principle in the closed loop, frequency-domain-based approaches for guaranteeing stability, which are popular design tools for LTI systems, can no longer directly and non-conservatively be applied. Thus, alternative methods based on constructing common quadratic or continuous piecewise quadratic (CPQ) Lyapunov functions [10] were developed for guaranteeing input-to-state stability (ISS) of HIGS-controlled systems, see, e.g., [5, 11]. In these methods, a suitable set of linear matrix inequalities (LMIs) are defined such that a solution to the LMIs, which can be found by semidefinite programming (SDP), indicates the existence of a Lyapunov function that proves ISS for the system. It was also shown that these methods could be used to calculate upper bounds on the H_2-norm and L_2-gain of conewise linear systems, which are commonly used performance metrics [11, 12].

In this paper, we are interested in extending the toolkit for analyzing the stability of conewise linear systems (CLS) in generic and HIGS-controlled systems in particular, without the need for SDP. Specifically, we are studying methods that use linear programming (LP) to construct continuous piecewise affine (CPA) Lyapunov functions. Although not used for HIGS and its variations so far, these methods have already been well discussed in the literature and several ways of formulating the LP problem have been proposed, see, e.g., [13–27]. These existing results motivate to study this CPA method for the specific class of HIGS-controlled systems, which we will do in this paper.

We present the following main contributions: Firstly, we present a theorem that applies the CPA method to conewise linear systems for the construction of ISS Lyapunov functions. The theorem is a modification of the CPA method from [27] with regional relaxations in the conditions of the LP problem constructed with Farkas' lemma. Secondly, we show that the CPA method can also be used to calculate upper bounds on important performance metrics, specifically, the L_1-gain and H_1-norm. Thirdly, a converse theorem is presented, which shows that given a stable conewise linear system and a sufficiently refined CPA function, the method presented in this paper always succeeds in finding a CPA Lyapunov function for the system. Finally, the proposed method will be demonstrated and compared to the CPQ method using LMIs for computational efficiency in a numerical case study. Part of these results were previously presented in a preliminary paper [28], on which we expand in this paper.

The remainder of this paper is structured as follows. Section 2 introduces notations and definitions used throughout this paper. In Section 3 HIGS and HIGS-controlled systems are discussed. Section 4 presents the modified CPA method for the stability analysis of conewise linear systems and for the calculation of upper bounds on the L_1-gain and H_1-norm. Subsequently, a converse theorem for the presented method is provided in Section 5. In Section 6 the CPA method will be compared to the CPQ method in a numerical case study. Finally, concluding remarks are given in Section 7.

2. Notation and definitions

In this paper, we will denote the subset of $\mathbb{R}$ with nonnegative entries with $\mathbb{R}_{\geq 0}$. Inequalities for vectors hold componentwise. The p-norm, where $p \geq 1$ and $p \in \mathbb{N}$, of a vector $x \in \mathbb{R}^n$ is defined as $\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$, where x_i is the ith component of x, and the infinity-norm as $\|x\|_\infty := \max_{i=1,2,\dots,n} |x_i|$. Unless otherwise specified, $\|\cdot\|$ can be an arbitrary norm. Finally, $\|A\|_p := \max_{\|x\|_p=1} \|Ax\|$ denotes the matrix norm of the matrix A induced by the p-norm.

Definition 1. *A signal $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is piecewise continuous (denoted by $x \in PC$), if there exists $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_{\geq 0}$, where $t_0 = 0$, $t_{k+1} > t_k$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$, such that x is continuous for all $t \in \mathbb{R}_{\geq 0} \setminus \{t_k\}_{k\in\mathbb{N}}$ and $\lim_{t\to t_k^+} x(t) = x(t_k)$. If $\int_0^\infty \|x(t)\|_1 \, dt < \infty$ in addition to $x \in PC$, then we write $x \in PC_1$.*

Consider the conewise linear system (CLS)

$$\begin{aligned} \dot{x}(t) &= A_j x(t) + B_j w(t) \quad \text{if } x(t) \in \mathcal{X}_j \\ y(t) &= C_j x(t) + D_j w(t) \end{aligned} \tag{1}$$

with state $x(t) \in \mathbb{R}^n$, disturbances $w(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$, all at time $t \in \mathbb{R}_{\geq 0}$. A_j, B_j, C_j and D_j are real matrices of appropriate dimensions, $\mathcal{X}_j$ are non-overlapping subsets of $\mathbb{R}^n$, where $j \in \{1, ..., M\}$ and $M \in \mathbb{N}$, and $\mathcal{X} := \bigcup_{j=1}^M \mathcal{X}_j \subseteq \mathbb{R}^n$. The closure of $\mathcal{X}_j$ is described by either a polyhedral cone or the intersection of a polyhedral cone and a hyperplane, i.e.,

$$\bar{\mathcal{X}}_j = \{x \in \mathbb{R}^n \mid F_j x \geq 0 \wedge \Pi_j x = 0\}, \tag{2}$$

where $\bar{\mathcal{X}}_j$ denotes the closure of a set $\mathcal{X}_j$, $F_j \in \mathbb{R}^{q\times n}$ is a matrix with $q \leq n$ and $\Pi_j^\top \in \mathbb{R}^n$ is an n-dimensional row vector. The right-hand side of (1) is possibly discontinuous.

The signal $x : \mathbb{R}_{\geq 0} \to \mathcal{X}$ is a solution to (1) in the sense of Carathéodory, if x is absolutely continuous and (1) is satisfied for almost all $t \in \mathbb{R}_{\geq 0}$ [29]. We assume existence and forward completeness of solutions, which for HIGS-controlled systems, introduced below, was shown in [6] if $w \in PC$. The derivative of a solution x can be ill-defined and, therefore, the Dini-derivative is used instead in those cases, which is defined as

$$D^+x(t) = \limsup_{h\to 0^+} \frac{x(t+h)-x(t)}{h}.$$

To make the notion of stability of discontinuous systems precise, we introduce the following definitions and theorem from [30], which are based on the work of [31].

Definition 2. *System* (1) *is said to be input-to-state stable (ISS), if there exist a $\mathcal{KL}$-function*[2] *β and a $\mathcal{K}$-function γ, such that for any initial state $x(0) \in \mathcal{X}$ and any bounded input $w \in PC$, any corresponding solution x of* (1) *satisfies*

$$\left\|x(t)\right\| \leq \beta\left(\left\|x(0)\right\|, t\right) + \gamma\left(\sup_{0\leq\tau\leq t}\left\|w(\tau)\right\|\right) \quad \forall t \in \mathbb{R}_{\geq 0}.$$

Definition 3. *A locally Lipschitz continuous function $V\colon \mathbb{R}^n \to \mathbb{R}$ is said to be an ISS Lyapunov function for system* (1)*, if it satisfies for all solutions x and corresponding $w \in PC$*

$$\alpha_1(||x(t)||) \leq V(x(t)) \leq \alpha_2(||x(t)||) \quad \textit{and} \tag{3a}$$
$$D^+V(x(t)) \leq -\alpha_3(||x(t)||) + \alpha_4(||w(t)||) \tag{3b}$$

for all $t \in \mathbb{R}_{\geq 0}$, where α_1, α_2 and α_3 are $\mathcal{K}_\infty$-functions and α_4 is a $\mathcal{K}$-function.

Theorem 1. *If there exists an ISS Lyapunov function for* (1)*, then* (1) *is ISS.*

[2]We adopt the standard definitions for $\mathcal{K}$-, $\mathcal{K}_\infty$- and $\mathcal{KL}$-functions from Definitions 4.2 and 4.3 in [32].

Finally, consider the following performance metrics.

Definition 4. *[33] The L_1-gain, i.e., the L_1-induced norm, of* (1) *is defined as the infimal value of γ for which*

$$\int_0^\infty \left\|y(t)\right\|_1 dt \leq \gamma \int_0^\infty \left\|w(t)\right\|_1 dt$$

for all $w \in PC_1$, $x(0) = 0$ and any corresponding output y.

Definition 5. *The H_1-norm of system* (1) *corresponding to an initial value $x(0) = x_0$ is defined as the infimal value of δ for which*

$$\int_0^\infty \left\|y(t)\right\|_1 dt \leq \delta,$$

for any output y given initial condition $x(0) = x_0$ and $w(t) = 0$ for all $t \in \mathbb{R}_{\geq 0}$.

The H_1-norm is defined in line with the H_2-norm as defined for certain classes of hybrid systems in [11, 12].

3. HIGS and HIGS-controlled systems

In continuous time, HIGS [4, 5] is described by

$$\mathcal{H} : \begin{cases} \dot{x}_h(t) = \omega_h e(t) & \text{if } \ (e(t), u(t), \dot{e}(t)) \in \mathcal{F}_1 & \text{(4a)} \\ x_h(t) = k_h e(t) & \text{if } \ (e(t), u(t), \dot{e}(t)) \in \mathcal{F}_2 & \text{(4b)} \\ u(t) \ = x_h(t) \end{cases}$$

with state $x_h(t) \in \mathbb{R}$, error $e(t) \in \mathbb{R}$ and its derivative $\dot{e}(t) \in \mathbb{R}$, control action $u(t) \in \mathbb{R}$, all at time $t \in \mathbb{R}_{\geq 0}$. The design parameters ω_h, $k_h \in [0, \infty)$ are, respectively, the integrator frequency and the gain. Eq. (4a) is called the integrator mode and (4b) the gain mode. The subsets $\mathcal{F}_1, \mathcal{F}_2 \subset \mathbb{R}^n$ are visualized in Fig. 1 and are defined by

$$\begin{aligned} \mathcal{F} \ &:= \mathcal{F}_1 \cup \mathcal{F}_2 = \{(e, u, \dot{e}) \in \mathbb{R}^3 \mid k_h e u \geq u^2\} \\ \mathcal{F}_1 &:= \mathcal{F} \backslash \mathcal{F}_2 \\ \mathcal{F}_2 &:= \{(e, u, \dot{e}) \in \mathbb{R}^3 \mid u = k_h e \wedge \omega_h e^2 > k_h e \dot{e}\}, \end{aligned}$$

where we drop the dependence on t for shortness where possible.

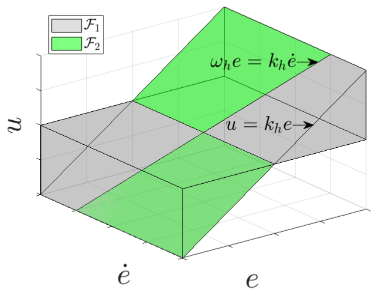

Fig. 1: Subsets $\mathcal{F}_1$, $\mathcal{F}_2$ in $(e, u, \dot{e})$-space [4].

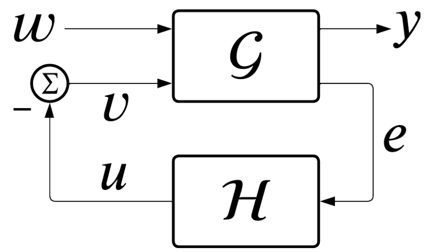

Fig. 2: LTI system $\mathcal{G}$ interconnected in closed loop with HIGS $\mathcal{H}$.

HIGS can be interpreted as follows: Assuming that $(e(0), u(0), \dot{e}(0)) \in \mathcal{F}$ at initial time $t = 0$, e.g., if $x_h(0) = 0$, the integrator mode will standardly be active. However, if the integrator dynamics would cause $(e, u, \dot{e})$ to leave the set $\mathcal{F}$, then the gain mode is momentarily activated. This ensures that $(e, u, \dot{e})$ lies in $\mathcal{F}$ for all $t \in \mathbb{R}_{\geq 0}$. See [4, 5] for more details.

The HIGS above is placed in a negative feedback loop, as shown in Fig. 2, with the LTI system $\mathcal{G}$ given by

$$\mathcal{G} : \begin{cases} \dot{x}_g(t) &= A_g x_g(t) + B_{gv} v(t) + B_{gw} w(t) \\ e(t) &= C_{ge} x_g(t) + D_{gev} v(t) + D_{gew} w(t) \\ y(t) &= C_{gy} x_g(t) + D_{gyv} v(t) + D_{gyw} w(t) \end{cases}$$

with state $x_g(t) \in \mathbb{R}^{n_g}$, control action $v(t) \in \mathbb{R}$, disturbances $w(t) \in \mathbb{R}^m$, error $e(t) \in \mathbb{R}$ and output $y(t) \in \mathbb{R}^p$, all at time $t \in \mathbb{R}_{\geq 0}$. A_g, B_{gv}, B_{gw}, C_{ge}, C_{gy}, D_{gev}, D_{gyv}, D_{gew} and D_{gyw} are real matrices of appropriate dimensions. The system $\mathcal{G}$ includes plant dynamics and potentially LTI controllers present in the control loop. Provided that the transformations from v and w to e in $\mathcal{G}$ have relative degree two or higher, i.e., $D_{ev} = C_{ge} B_{gv} = 0$ and $D_{ew} = C_{ge} B_{gw} = 0$, the HIGS-controlled system can be described by

$$\begin{aligned} \dot{x}(t) &= \begin{cases} A_1 x(t) + B w(t) & \text{if } Hx(t) \in \mathcal{F}_1 \\ A_2 x(t) + B w(t) & \text{if } Hx(t) \in \mathcal{F}_2 \end{cases} \\ y(t) &= C x(t) + D w(t), \end{aligned} \tag{5}$$

where $x = \begin{bmatrix} x_g^\top & x_h \end{bmatrix}^\top \in \mathbb{R}^n$ with $n = n_g + 1$,

$$\left[\begin{array}{c|c} A_k & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A_g & -B_{gv} & B_{gw} \\ A_{h,k} & 0 & 0_{1\times m} \\ \hline C_{gy} & -D_{gyv} & D_{gyw} \end{array}\right],$$

$$A_{h,1} = \omega_h C_{ge}, \quad A_{h,2} = k_h C_{ge} A_g$$

and

$$H = \begin{bmatrix} C_{ge} & 0 \\ 0_{1\times n_g} & 1 \\ C_{ge}A_g & 0 \end{bmatrix}.$$

Note that (5) can be written as a conewise linear system (1), since regions $\mathcal{F}_1$ and $\mathcal{F}_2$ can be partitioned into subregions $\mathcal{X}_j$ that satisfy (2).

4. Stability and performance analysis with linear programming

In order to parameterize a CPA function, we make use of a triangulation of the domain of the state space. The specifics of such a triangulation will be discussed first in this section. Following that, a theorem will be presented that provides sufficient conditions under which the CPA Lyapunov function is an ISS Lyapunov function to a CLS. Additionally, it will be shown that the theorem can also be used to determine upper bounds on the L_1-gain and the H_1-norm of a CLS.

4.1. The triangulation

We define a triangulation $\mathcal{T}$ as a set of simplicial cones $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_N$, where $N \in \mathbb{N}$. A simplicial cone $\mathcal{S}_i \subset \mathbb{R}^n$, $i \in \{1, 2, ..., N\}$, is the positive hull of n linearly independent vertices $r_1^i, ..., r_n^i$, i.e.,

$$\begin{aligned} \mathcal{S}_i &= \left\{ x \in \mathbb{R}^n \mid x = \sum_{k=1}^{n} \lambda_k r_k^i \text{ for some } \lambda_k \geq 0,\, k \in \{1, 2, ..., n\} \right\} \\ &= \{x \in \mathbb{R}^n \mid E_i x \geq 0\}, \end{aligned}$$

where $E_i := \begin{bmatrix} r_1^i & \dots & r_n^i \end{bmatrix}^{-1}$.

Let $\mathcal{X} := \bigcup_{j=1}^{M} \mathcal{X}_j$ be the domain of the state x of system (1). Then, a triangulation $\mathcal{T}$ of $\mathcal{X}$ is a set of simplicial cones $\{\mathcal{S}_1, ..., \mathcal{S}_N\}$ such that

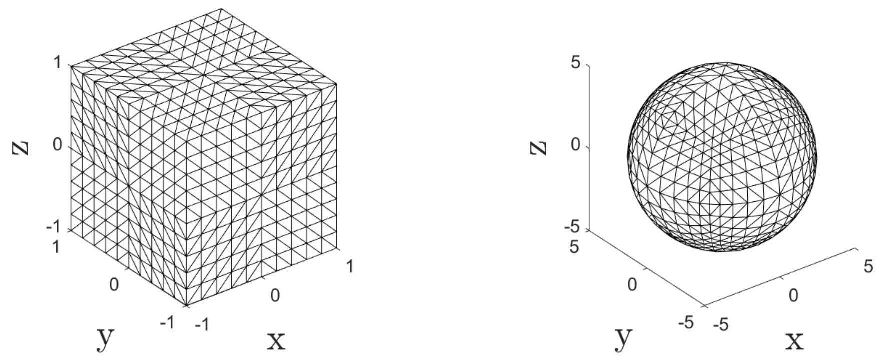

Fig. 3: Visualization of the triangulation $K^{-1}\mathcal{T}_5$ (left) and $\mathcal{T}_5^F$ (right) for $n = 3$. Each triangle represents a cross section of a simplicial cone and the corners of the triangles represent the vertices of the triangulation.

$\mathcal{X} = \bigcup_{i=1}^{N} \mathcal{S}_i$. The triangulation $\mathcal{T}$ is shape-regular if all $\mathcal{S}_i$ and $\mathcal{S}_j$, where $i, j \in \{1, ..., N\}$ and $i \neq j$, either intersect in a common, potentially lower dimensional, face or not at all.

Examples of shape-regular triangulations are the triangulations $K^{-1}\mathcal{T}_K$ and $\mathcal{T}_K^F$ from [34], where $K \in \mathbb{N}$ is a measure of how refined the triangulation is. For example, the triangulation $K^{-1}\mathcal{T}_K$ has $N = 2^n \cdot K^{n-1} \cdot n!$ simplicial cones and vertices that satisfy $\left\|r_k^i\right\|_\infty = 1$ and $\left\|r_k^i - r_l^i\right\|_2 \leq \sqrt{n}/K$ for all $i \in \{1, 2, ..., N\}$ and all $k, l \in \{1, 2, ..., n\}$. The triangulation $\mathcal{T}_K^F$ scales the vertices of $K^{-1}\mathcal{T}_K$ with a factor $K/\left\|r_k^i\right\|_2$ for numerical conditioning. We will use the triangulation $K^{-1}\mathcal{T}_K$ for the converse theorem in Section 5 and the triangulation $\mathcal{T}_K^F$ for the numerical case study in Section 6. A visualization of the two triangulations is given in Fig. 3.

4.2. Construction of CPA Lyapunov functions

We parameterize a CPA function using a triangulation $\mathcal{T}$ of $\mathcal{X}$ by assigning a value V_r to each unique vertex r of a simplicial cone in $\mathcal{T}$, i.e., if $r_k^i = r_l^j$ for some $i, j \in \{1, 2, ..., N\}$ and $k, l \in \{1, 2, ..., n\}$, then $V_{r_k^i} = V_{r_l^j}$. Then, a CPA Lyapunov function candidate can be described by

$$V(x) = v_i^\top E_i x \quad \text{if } x \in \mathcal{S}_i, \tag{6}$$

where $v_i^\top = \begin{bmatrix} V_{r_1^i} & \dots & V_{r_n^i} \end{bmatrix}$, $i \in \{1, ..., N\}$, and the dependency of $x(t)$ (and $w(t)$ and $y(t)$) on t is from here on omitted for brevity where possible. Note that V is positively homogeneous of degree one, i.e., $V(cx) = cV(x)$ for all positive c, V is locally Lipschitz continuous by construction, $V(r_k^i) = V_{r_k^i}$ for all $k \in \{1, ..., n\}$ and $i \in \{1, ..., N\}$, and the value of $V(x)$ in (6) is found by linearly interpolating between the values of $V(r_k^i)$ and $V(0) = 0$ if $x \in \mathcal{S}_i$.

The following theorem poses sufficient conditions under which (6) is an ISS Lyapunov function to (1).

Theorem 2. *Consider system* (1) *and a shape-regular triangulation $\mathcal{T}$ of $\mathcal{X}$. Suppose there exist variables $V_r \in \mathbb{R}$, nonnegative vectors $u_{ij} \in \mathbb{R}^n_{\geq 0}$, real variables $l_{ij} \in \mathbb{R}$, where $i \in \{1, ..., N\}$ and $j \in \{1, ..., M\}$, and constants $\epsilon_1, \epsilon_2 > 0$ such that:*

1. *for every vertex r of a simplicial cone in $\mathcal{T}$*

$$V_r \geq \epsilon_1 \|r\|\,;$$

2. *for all $i \in \{1, ..., N\}$, $j \in \{1, ..., M\}$ and $k \in \{1, ..., n\}$*

$$(v_i^\top E_i A_j + u_{ij}^\top F_j + l_{ij} \Pi_j) r_k^i \leq -\epsilon_2 \left\| r_k^i \right\| - \left\| C_j r_k^i \right\|_1 .$$

Then (6) *is an ISS Lyapunov function of* (1)*. If, in addition to conditions* 1 *and* 2*, there exists a variable $\gamma \in \mathbb{R}$ such that:*

3. *for all $i \in \{1, ..., N\}$, $j \in \{1, ..., M\}$ and $l \in \{1, ..., m\}$*

$$\pm v_i^\top E_i B_j e_l \leq \gamma \|e_l\|_1 - \left\| D_j e_l \right\|_1 ,$$

   where e_l is the lth standard unit vector of $\mathbb{R}^m$,

then γ is an upper bound on the L_1-gain of (1)*. If, in addition to conditions* 1 *and* 2*, there exists a variable $\delta \in \mathbb{R}$ such that:*

4. *given $x_0 \in \mathcal{S}_q$*

$$v_q^\top E_q x_0 \leq \delta,$$

then δ is an upper bound on the H_1-norm corresponding to an initial condition $x(0) = x_0$ of (1)*.*

Proof. Firstly, we will prove that condition 1 implies that V satisfies (3a). Because of the triangle inequality, V satisfies

$$\begin{aligned} V(x) &= v_i^\top E_i x = v_i^\top E_i \sum_{k=1}^{n} \lambda_k r_k^i = \sum_{k=1}^{n} \lambda_k (v_i^\top E_i r_k^i) \\ &= \sum_{k=1}^{n} \lambda_k V_{r_k^i} \geq \sum_{k=1}^{n} \lambda_k \epsilon_1 \left\| r_k^i \right\| = \epsilon_1 \sum_{k=1}^{n} \lambda_k \left\| r_k^i \right\| \\ &\geq \epsilon_1 \left\| \sum_{k=1}^{n} \lambda_k r_k^i \right\| = \epsilon_1 \|x\| \,. \end{aligned}$$

Moreover, from the Cauchy-Schwarz inequality and the equivalence of norms it follows that

$$V(x) = v_i^\top E_i x \leq \left\| E_i^\top v_i \right\|_2 \|x\|_2 \leq c_1 \|x\| \tag{7}$$

for some positive constant c_1. Thus, V satisfies (3a).

Secondly, we will prove that condition 2 implies that V satisfies (3b). Note that, if $x(t) \in \mathcal{S}_i$ and $x(t+\delta t) \in \mathcal{S}_i$ for an infinitesimal small time step δt, then

$$D^+ V(x) = v_i^\top E_i (A_j x + B_j w) \qquad \forall x \in \mathcal{X}_j \cap \mathcal{S}_i.$$

Thus, V satisfies condition (3b), if

$$v_i^\top E_i (A_j x + B_j w) \leq -c_2 \|x\| + c_3 \|w\| \quad \forall x \in \bar{\mathcal{X}}_j \cap \mathcal{S}_i$$

for some $c_2, c_3 > 0$ and all $w \in \mathbb{R}^m$. By Farkas' lemma and the Fredholm alternative, this is equivalent to

$$v_i^\top E_i A_j x + u_{ij}^\top F_j x + l_{ij} \Pi_j x \leq -c_2 \|x\| \quad \forall x \in \mathcal{S}_i, \tag{8}$$

where F_j and Π_j are defined in (2). Here, we make use of the fact that $v_i^\top E_i B_j w$ is always smaller than $c_3 \|w\|$ for some positive constant c_3, see, e.g., (7). Thus, by proving (8), we prove that V satisfies condition (3b). Since $x \in \mathcal{S}_i$, $x = \sum_{k=1}^{n} \lambda_k r_k^i$ for some $\lambda_k \geq 0$, $k \in \{1, 2, ..., n\}$. Thus, condition 2 implies that

$$\begin{aligned}
&(v_i^\top E_i A_j + u_{ij}^\top F_j + l_{ij}\Pi_j)x \\
=&(v_i^\top E_i A_j + u_{ij}^\top F_j + l_{ij}\Pi_j)\sum_{k=1}^{n}\lambda_k r_k^i \\
=&\sum_{k=1}^{n}\lambda_k (v_i^\top E_i A_j + u_{ij}^\top F_j + l_{ij}\Pi_j) r_k^i \\
\leq&\sum_{k=1}^{n}\lambda_k \left(-\epsilon_2\left\|r_k^i\right\| - \left\|C_j r_k^i\right\|_1\right) \\
\leq&-\epsilon_2\|x\| - \left\|C_j x\right\|_1 .
\end{aligned} \tag{9}$$

Eq. (9) proves (8) and therefore V satisfies (3b) and V is an ISS Lyapunov function of (1).

Thirdly, it is shown that γ is an upperbound on the L_1-gain due to conditions 1-3. Consider that $w = \sum_{l=1}^{m}\mu_l e_l$ for some real variables μ_l. Then, condition 3 implies that for all $i \in \{1, 2, ..., N\}$ and $j \in \{1, 2, ..., M\}$

$$\begin{aligned}
&v_i^\top E_i B_j w = v_i^\top E_i B_j \sum_{l=1}^{m}\mu_l e_l = \sum_{l=1}^{m}\mu_l v_i^\top E_i B_j e_l \\
\leq&\sum_{l=1}^{m}|\mu_l|\left(\gamma\|e_l\|_1 - \left\|D_j e_l\right\|_1\right) \leq -\left\|D_j w\right\|_1 + \gamma\|w\|_1 .
\end{aligned}$$

By combining this result with (9), we obtain that, if $x \in \bar{\mathcal{X}}_j \cap \mathcal{S}_i$,

$$\begin{aligned}
D^+V(x) = v_i^\top E_i(A_j x + B_j w) &\leq (v_i^\top E_i A_j + u_{ij}^\top F_j + l_{ij}\Pi_j)x + v_i^\top E_i B_j w \\
&\leq -\epsilon_2\|x\| - \left\|C_j x\right\|_1 - \left\|D_j w\right\|_1 + \gamma\|w\|_1 \\
&\leq -\epsilon_2\|x\| - \|y\|_1 + \gamma\|w\|_1 .
\end{aligned} \tag{10}$$

By integrating both sides of (10) from 0 to $T > 0$, we obtain that

$$\begin{aligned}
&\int_0^T D^+V(x(t))\mathrm{d}t = V(x(T)) - V(x(0)) \\
\leq&-\epsilon_2\int_0^T \left\|x(t)\right\|\mathrm{d}t - \int_0^T \left\|y(t)\right\|_1\mathrm{d}t + \gamma\int_0^T \left\|w(t)\right\|_1\mathrm{d}t.
\end{aligned} \tag{11}$$

Since V is a positive definite function, due to condition 1, and given $x(0) = 0$ (see Definition 4), this implies that

$$\gamma \int_0^T \left\| w(t) \right\|_1 \mathrm{d}t \geq \int_0^T \left\| y(t) \right\|_1 \mathrm{d}t$$

for all $T > 0$, which proves γ is an upper bound on the L_1-gain of (1) when $T \to \infty$.

Finally, we prove that conditions 1, 2 and 4 imply that δ is an upper bound on the H_1-norm corresponding to an initial condition $x(0) = x_0$. Since $w(t) = 0$ for all $t \in \mathbb{R}_{\geq 0}$ (see Definition 5) and thus $\lim_{T\to\infty} V(x(T)) = 0$, since the system is ISS if conditions 1 and 2 are met, it follows from condition 4 and (11) that

$$\delta \geq v_q^\top E_q x_0 = V(x(0)) \geq \int_0^\infty \left\| y(t) \right\|_1 ,$$

which proves δ is an upper bound on the H_1-norm corresponding to initial condition x_0 of (1). $\square$

The existence of suitable V_r, γ, δ, u_{ij} and l_{ij} for Theorem 2 can be verified by solving an LP feasibility problem with optimization variables V_r, γ, δ, u_{ij} and l_{ij}, constants ϵ_1 and ϵ_2 and linear constraints 1-4.

Remark 1. *Note that in the conditions for ISS, i.e., conditions 1 and 2 in Theorem 2, the matrices B_j and the disturbance w are not considered. This is because, for a CLS* (1)*, it is sufficient to prove global exponential stability of the origin, which implies ISS, as the inputs enter the dynamics in an affine manner.*

Remark 2. *Note that the HIGS-controlled system in* (5) *can have a discontinuous right-hand side. This can lead to robust stability issues [35]. Namely, given small perturbations of the state,* (5) *can become unstable, even though the nominal (unperturbed) system is stable. For robust stability guarantees, the stability of the Krasovskii regularization of* (5) *should be studied, which is defined as*

$$\begin{aligned}\dot{x}(t) \in &\begin{cases} A_1 x(t) + Bw(t) & \text{if } Hx(t) \in \mathcal{F}_1 \backslash \bar{\mathcal{F}}_2 \\ \mathrm{co}\{A_1 x(t), A_2 x(t)\} + Bw(t) & \text{if } Hx(t) \in \bar{\mathcal{F}}_2 \end{cases}\\ y(t) =& Cx(t) + Dw(t),\end{aligned}$$

where $\mathrm{co}\{\mathcal{F}\}$ *denotes the closed convex hull of a set* $\mathcal{F} \subset \mathbb{R}^n$*. It can be shown that the conditions in Theorem 2 can guarantee stability of the Krasovskii regularized system as well. See also [5, Remark 6.1].*

Remark 3. *Alternatively to using Farkas' lemma and the Fredholm alternative, one could choose the triangulation* $\mathcal{T}$ *such that the intersection of* $\mathcal{S}_i$ *and the closure of* $\mathcal{X}_j$ *is empty or equal to* $\mathcal{S}_i$ *or a lower dimensional face of* $\mathcal{S}_i$*, i.e.,*

$$\bar{\mathcal{X}}_j \cap \mathcal{S}_i = \left\{ x \in \mathbb{R}^n \mid x = \sum_{k=1}^{m} \lambda_k s_k^{ij}, \lambda_k \geq 0 \right\},$$

where $m \leq n$ *and* $\{s_1^{ij}, ..., s_m^{ij}\} \subseteq \{r_1^i, ..., r_n^i\}$*, for all* $i \in \{1, ..., N\}$ *and* $j \in \{1, ..., M\}$*. Then, condition 2 in Theorem 2 can be changed to*

2'. *for all* $i \in \{1, ..., N\}$*,* $j \in \{1, ..., M\}$ *and* $s \in \{s_1^{ij}, ..., s_m^{ij}\}$

$$v_i^\top E_i A_j s \leq -\epsilon_2 \|s\| - \left\| C_j s \right\|_1.$$

While this would reduce the number of optimization variables in the resulting LP problem, it also restricts our options for choosing a computationally efficient triangulation $\mathcal{T}$ *and, thus, may lead to potentially increased conservatism.*

5. Converse theorem

In this section we will show that our method always succeeds in finding a CPA Lyapunov function for an ISS conewise linear system (1) if the triangulation $K^{-1}\mathcal{T}_K$ has sufficiently high K, i.e., a sufficiently large number of thin simplicial cones.

Theorem 3. *Given an ISS conewise linear system* (1)*, the conditions 1 and 2 of Theorem 2 will always be satisfied when using the triangulation* $K^{-1}\mathcal{T}_K$ *for a sufficiently large* K *and sufficiently small, but positive,* ϵ_1 *and* ϵ_2*.*

Proof. Firstly, consider the differential inclusion

$$\dot{x}(t) \in \mathrm{co} \left\{ \bigcup_{j=1}^{M} \{A_j x(t) \mid x(t) \in \bar{\mathcal{X}}_j\} \right\} =: F(x(t)). \tag{12}$$

If (1) is ISS, the differential inclusion (12) must necessarily be strongly asymptotically stable [36] and therefore must admit an infinitely smooth

Lyapunov function. Since the differential inclusion (12) is positively homogeneous of degree 1, there also exists an infinitely smooth Lyapunov function $W : \mathbb{R}^n \to \mathbb{R}$ that is positively homogeneous of degree 1 [37]. Thus, W satisfies for some $c_1, c_2 > 0$

$$\begin{aligned}
W(cx) &= cW(x) && \forall c > 0, x \in \mathbb{R}^n \\
W(x) &= \nabla W(x) \cdot x && \forall x \in \mathbb{R}^n \\
W(x) &\geq c_1\|x\| && \forall x \in \mathbb{R}^n \\
W(0) &= 0 && \\
\sup_{v \in F(x)} \nabla W(x) v &\leq -c_2\|x\|_2 && \forall x \in \mathbb{R}^n.
\end{aligned} \tag{13}$$

Secondly, we assign $V_r = W(r)$ for every vertex r of a simplicial cone in the triangulation $K^{-1}\mathcal{T}_K$. Then, it follows from (13) that $V_r \geq \epsilon_1\|r\|$ for $\epsilon_1 \leq c_1$. Thus, conditions 1 in Theorem (2) are satisfied for sufficiently low ϵ_1.

Next, it is desirable to derive an upperbound on $\left\|v_i^\top E_i - \nabla W(x)\right\|_2$ for all $x \in \{y \in \mathcal{S}_i | \|y\|_\infty = 1\}$, $i \in \{1, 2, ..., N\}$. To this end, we follow similar steps as in [34]: Consider that

$$v_i^\top = \begin{bmatrix} V_{r_1^i} & ... & V_{r_n^i} \end{bmatrix} = \begin{bmatrix} W(r_1^i) & ... & W_(r_n^i) \end{bmatrix}$$

and, due to Taylor's theorem and (13), we know that for all $k \in \{1, 2, ..., n\}$

$$\begin{aligned}
W(r_k^i) =& W(x) + \nabla W(x) \cdot (r_k^i - x) + \frac{1}{2}(r_k^i - x)^\top H(\xi)(r_k^i - x) \\
=& \nabla W(x) \cdot r_k^i + \frac{1}{2}(r_k^i - x)^\top H(\xi)(r_k^i - x),
\end{aligned}$$

where $H(x)$ is the Hessian of $W(x)$ and ξ is a point on the line segment between x and r_k^i. Define $\tilde{H} := \max_{\|x\|_\infty = 1} \left\|H(x)\right\|_2$. Then,

$$\left|W(r_k^i) - \nabla W(x) \cdot r_k^i\right| \leq \tilde{H}\left\|r_k^i - x\right\|_2^2.$$

It was shown in [34] that $\left\|r_k^i - x\right\|_2 \leq \sqrt{n}/K$ for all $x \in \{y \in \mathcal{S}_i | \|y\|_\infty = 1\}$. Since $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$, this implies that

$$\left\|v_i^\top - \nabla W(x) \begin{bmatrix} r_1^i & ... & r_n^i \end{bmatrix}\right\|_2 \leq \sqrt{n}\tilde{H}\left(\frac{\sqrt{n}}{K}\right)^2 = \tilde{H}\frac{n\sqrt{n}}{K^2}.$$

Furthermore, it was shown in [34] that $\|E_i\|_2 \leq K\sqrt{3(n+1)}$. Thus,

$$\begin{aligned}\left\|v_i^\top E_i - \nabla W(x)\right\|_2 =& \left\|\left[v_i^\top - \nabla W(x)E_i^{-1}\right] E_i\right\|_2 \\ \leq& \left\|v_i^\top - \nabla W(x)\begin{bmatrix} r_1^i & \dots & r_n^i\end{bmatrix}\right\|_2 \|E_i\|_2 \\ \leq& \tilde{H}\frac{n\sqrt{n}}{K^2}K\sqrt{3(n+1)} = \frac{n\tilde{H}}{K}\sqrt{3n(n+1)}.\end{aligned}$$

With the upperbound we can show that for all $x \in \{y \in \bar{\mathcal{X}}_j \cap \mathcal{S}_i | \|y\|_\infty = 1\}$

$$\begin{aligned} v_i^\top E_i A_j x =& \nabla W(x) A_j x + [v_i^\top E_i - \nabla W(x)] A_j x \\ \leq& - c_2\|x\|_2 + n\sqrt{3n(n+1)}\tilde{H}K^{-1}\tilde{A}\|x\|_2 \\ =& - c_3\|x\|_2 \leq -c_3\|x\|_\infty \\ \leq& \pm\, c_3 e_k^\top x \qquad \forall k \in \{1,2,...,n\} \end{aligned} \tag{14}$$

where $\tilde{A} := \max_{j\in\{1,2,\ldots,M\}} \left\|A_j\right\|_2$ and $c_3 := c_2 - n\sqrt{3n(n+1)}\tilde{H}K^{-1}\tilde{A}$. For sufficiently large K, c_3 will be positive. Since (14) is positively homogeneous of degree 1, (14) holds for all $x \in \bar{\mathcal{X}}_j \cap \mathcal{S}_i$, i.e., $\|x\|_\infty = 1$ is not required.

Finally, since $x \in \bar{\mathcal{X}}_j \cap \mathcal{S}_i$ can be expressed as linear inequalities

$$\begin{bmatrix} F_j \\ \Pi_j \\ -\Pi_j \\ E_i \end{bmatrix} x =: Z_{ij} x \geq 0,$$

$v_i^\top E_i A_j \leq \pm c_3 e_k^\top x$ if $Z_{ij}x \geq 0$, for all $k \in \{1,2,...,n\}$, implies by Farkas' lemma that there exists $w_{ij} \in \mathbb{R}_{\geq 0}^{2n+2}$ such that for all $x \in \mathbb{R}^n$

$$v_i^\top E_i A_j x + w_{ij}^\top Z_{ij} x \leq \pm c_3 e_k^\top x \qquad \forall k \in \{1,2,...,n\}.$$

This implies that

$$v_i^\top E_i A_j x + w_{ij}^\top Z_{ij} x \leq -c_3\|x\|_\infty. \tag{15}$$

We substitute x with r_k^i in (15) and rewrite the resulting equation as

$$(v_i^\top E_i A_j + u_{ij}^\top F_{ij} + l_{ij}^\top \Pi_j) r_k^i + z_{ij}^\top E_i r_k^i \leq -c_3\left\|r_k^i\right\|_\infty, \tag{16}$$

where $u_{ij}, z_{ij} \in \mathbb{R}^n_{\geq 0}$ and $l_{ij} \in \mathbb{R}^n$. Note that $z_{ij}^\top E_i r_k^i$ is necessarily nonnegative and can thus be omitted. Thus, (16) implies

$$\begin{aligned}\frac{\epsilon_2 + \left\|C_j\right\|_\infty}{c_3}(v_i^\top E_i A_j + u_{ij}^\top F_{ij} + l_{ij}^\top \Pi_j) r_k^i &\leq -\left(\epsilon_2 + \left\|C_j\right\|_\infty\right)\left\|r_k^i\right\|_\infty \\ &\leq -\epsilon_2 \left\|r_k^i\right\|_\infty - \left\|C_j r_k^i\right\|_\infty .\end{aligned}$$

Note that $\left(\epsilon_2 + \left\|C_j\right\|_\infty\right)/c_3$ is an arbitrary positive constant that can be incorporated into the optimization variables v_i, u_{ij} and l_{ij}, and thus, by the equivalence of norms, is equivalent to condition 2 of Theorem 2. Thus, for sufficiently high K and low ϵ_2 conditions 2 of Theorem 2 are satisfied. $\square$

Remark 4. *While Theorem 3 proves that conditions 1 and 2 of Theorem 2 are satisfied for a sufficiently large K of the triangulation $K^{-1}\mathcal{T}_K^F$, it does not prove satisfaction of conditions 3 and 4 of Theorem 2. Whether the upperbounds on the L_1-gain and H_1-norm determined by Theorem 2 are conservative in the limit of K thus remains an open question.*

6. Numerical case study

We apply the presented CPA method to the stability and performance analysis of a HIGS-controlled system. Additionally, we compare the CPA method to the method in [11], henceforth referred to as the CPQ method, which is based on the work in [10]. This method involves verifying the feasibility of a suitable set of linear matrix inequalities (LMIs), which, if successful, proves the existence of a continuous piecewise quadratic (CPQ) Lyapunov function. CPQ Lyapunov functions are parameterized with a triangulation $\mathcal{T}$ in an analogous way to the CPA Lyapunov functions in Section 4. The software package Mosek [38] implemented in MATLAB with Yalmip [39] is used to solve the LP and semidefinite programming (SDP) problems.

6.1. System description

Consider the LTI mass-spring-damper system

$$\mathcal{G}:\begin{cases}\dot{x}_g(t) &= \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} x_g(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) \\ e(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_g(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_g(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \end{cases}$$

where $m = b = k = 1$. We place the LTI system in a negative feedback loop with HIGS as shown in Fig. 2. The resulting HIGS-controlled system can be described by (5) or, equivalently, as a conewise linear system (1).

It was shown in [27] that it can be beneficial to the CPA method to introduce state transformations that cause the level sets of Lyapunov functions to resemble hyperspheres. This can reduce the number of simplicial cones N that the triangulation $\mathcal{T}_K^F$ and the CPA Lyapunov function V require. For HIGS-controlled systems specifically, we can exploit the fact that by design the integrator mode is the dominant mode of HIGS. Thus, to make the level sets of the Lyapunov functions resemble hyperspheres, we can introduce the state transformation

$$\hat{x} = V^{-1}x, \tag{17}$$

where the columns of V are the real and, if applicable, imaginary parts of the normalized eigenvectors of A_1, not counting complex conjugates.

6.2. Stability analysis

Using the CPA and CPQ method, we check whether the HIGS-controlled system is ISS for various combinations of k_h and ω_h. From Theorem 3, we know that for sufficiently large K a CPA Lyapunov function must exist. Similar results exist for CPQ Lyapunov functions [40]. Thus, both methods are non-conservative in the limit of K. However, increasing K also increases the number of constraints and optimization variables of the LP and SDP problems, which increases the computation time required for solving the feasibility problem. Therefore, we limit the available computation time of both methods to approximately 10^3 s. Additionally, we use extensive time-series simulations to determine which combinations (k_h, ω_h) are ISS to serve as a benchmark. See Fig. 4 for the results.

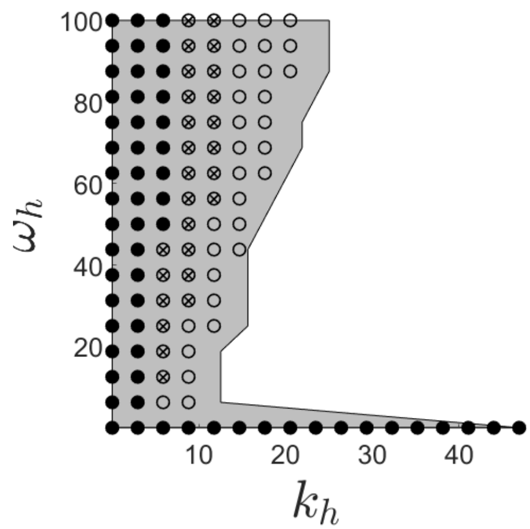

Fig. 4: ISS region found using time-series simulations (grey), ISS combinations (k_h, ω_h) found using the CPA method with triangulation $\mathcal{T}_5^F$ (black dots), ISS combinations (k_h, ω_h) found using the CPA method with triangulation $\mathcal{T}_5^F$ and state transformation (17) (black dots and crosses) and ISS combinations (k_h, ω_h) found using the CPQ method with triangulation $\mathcal{T}_2^F$ (black dots, crosses and circles).

Firstly, note that the state transformation (17) indeed has a positive effect on the accuracy of the CPA method as more stable ISS combinations (k_h, ω_h) are found when using the state transformation. Secondly, notice that the CPA method uses the triangulation $\mathcal{T}_5^F$, which has $N = 1200$ simplicial cones, and the CPQ method uses the triangulation $\mathcal{T}_2^F$, which has $N = 192$ simplicial cones. This is because the CPQ method requires more optimization variables and constraints than the CPA method and solving an SDP problem is in general slower than solving an LP problem. Thus, given the limited computation time of $\sim 10^3$ s, the CPQ method requires a triangulation with less simplicial cones than the CPA method. Nevertheless, the CPQ method can verify the stability of more combinations (k_h, ω_h) than the CPA method in the available computation time. This can be explained as follows: Both the CPA and CPQ method can be interpreted as trying to approximate the level set $W(x) = 1$ of an infinitely smooth Lyapunov function W that is homogeneous of arbitrary degree. If $n = 2$, the CPA method tries to approximate the level set with an $N-$polygon, i.e., with N linear segments, whereas the CPQ method uses N elliptic arcs to approximate the level set.

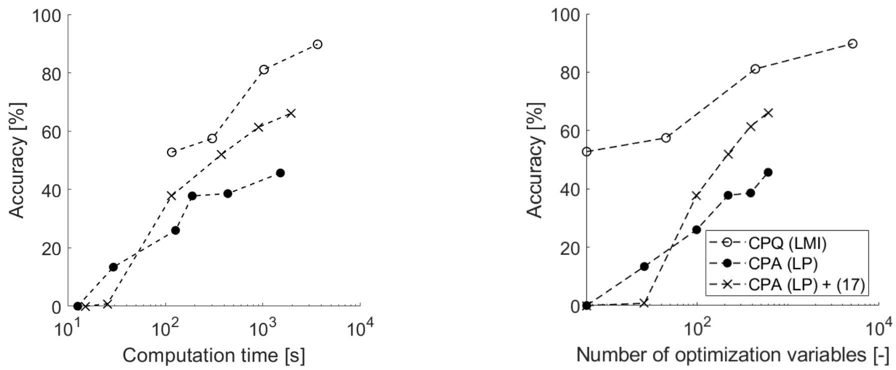

Fig. 5: Accuracy of the CPQ method, CPA method and CPA method with state transformation (17) for various triangulations versus the required computation time and the used number of optimization variables (not including optimization variables in regional relaxations).

An elliptic arc can approximate a line over a finite domain arbitrarily closely, but the reverse is not true. Therefore, the CPQ method will theoretically always require less piecewise segments N to closely approximate the level set of W than the CPA method. Thus, CPQ Lyapunov functions require a less dense triangulation for verifying stability. In the example, this property outweighs the computational benefits of LP over SDP.

We repeat this experiment for triangulations with varying refinements and note for each method and triangulation the computation time, number of optimization variables and the accuracy, defined as the percentage of stable ISS combinations found. See Fig. 5. This figure confirms the conclusions drawn above: Firstly, that the state transformation in (17) generally improves the accuracy of the CPA method and secondly, that the CPA method requires more computation time and more optimization variables to verify the stability of the same number of ISS combinations than the CPQ method.

Some comments should be made about these results. Firstly, when removing the restriction on computation time, it was found that the CPA method could find the same ISS combinations (k_h, ω_h) as the CPQ method. This is a necessary consequence of the converse theorem in Section 5. However, the CPA method required much more computation time than the CPQ method to do so. Secondly, in realistic industrial applications, the damping is typically much lower and the size of the HIGS-controlled system much larger than in the use case considered here. See, e.g., the case in [7]. When study-

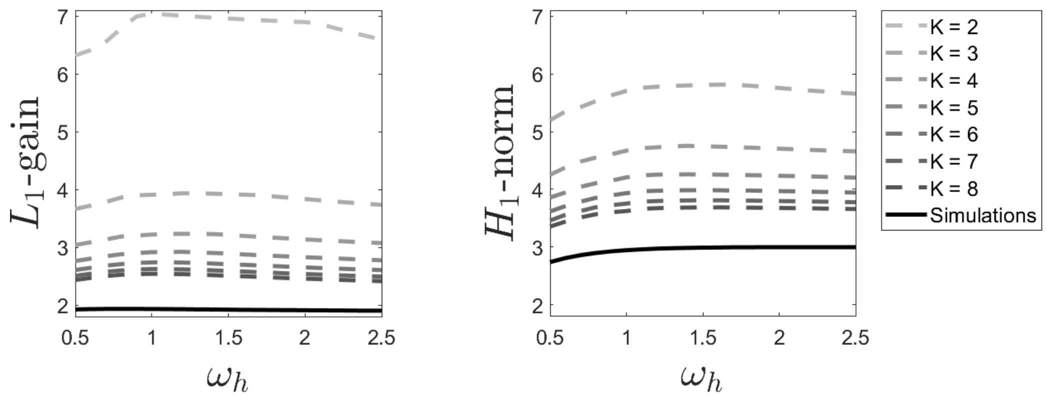

Fig. 6: Upper bound on L_1-gain and H_1-norm calculated using Theorem 2 for various triangulations $\mathcal{T}_K^F$ (grey). Additionally, using time-series simulations, a lower bound on the L_1-gain and an estimate of the H_1-norm are calculated (black).

ing such systems, it was found that CPA method performed worse than the CPQ method and the gap in accuracy between the CPA and CPQ methods increased when the damping was lowered and/or when the size of the HIGS-controlled system increased. Thirdly, while in the examples considered in this project the efficiency of CPQ Lyapunov functions over CPA Lyapunov functions outweighed the computational benefits of LP over SDP, this may not be the case for all systems.

6.3. Performance analysis

We apply Theorem 2 to calculate upper bounds on the L_1-gain and H_1-norm of the HIGS-controlled system. We fix the value of k_h to 1 and vary the value of ω_h over $[0.5, 2.5]$. For different triangulations $\mathcal{T}_K^F$ we calculate the upper bounds on the L_1-gain and the H_1-norm, corresponding to an initial condition

$$x(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top,$$

using Theorem 2. See Fig. 6.

In this figure, a numerical estimate of the H_1-norm and a numerical estimate of a lower bound on the L_1-gain determined using time-series simulations is also provided. The upper bound on the L_1-gain and H_1-norm becomes increasingly tight as we increase the value of K of the triangulation $\mathcal{T}_K^F$ used to parametrize the CPA Lyapunov function. Potential reasons for the remaining difference between the calculated upper bounds and the results from the time-series simulations are conservatism in Theorem 2, numerical artifacts and/or limited computational power.

7. Conclusion

For the stability and performance analysis of HIGS-controlled systems, we have investigated a linear programming-based approach for the construction of CPA Lyapunov functions as a possible alternative to the well-known CPQ method based on semidefinite programming. To this end, we have extended the approach to be applicable to HIGS-controlled systems and other conewise linear systems. This extension was presented in the form of a theorem that poses sufficient conditions for the existence of a CPA Lyapunov function that can be verified by linear programming. Additionally, it was shown that the conditions of this theorem become necessary provided that a CPA function with a sufficiently large number of thin piecewise segments is used. The extended framework can also be used to calculate an upper bound on the L_1-gain and H_1-norm of the system, which can be useful performance metrics. This method of constructing CPA Lyapunov functions was applied to the stability and performance analysis of a HIGS-controlled system. While the numerical example showed that this LP-based method was effective, it also showed that the method was less efficient in terms of computation time than the pre-existing SDP-based approach that constructs CPQ Lyapunov functions. This was mainly due to the large number of piecewise segments required for the construction of CPA Lyapunov functions. Therefore, future work will focus on seeking methods to reduce this number of segments. Moreover, we will investigate efficient LP-based methods to construct CPQ Lyapunov functions, given that CPQ functions showed good approximation properties in the example and that LP problems are generally faster and more robust to solve than SDP problems. Thus, seeking for such methods, possibly inspired by the improvement of existing LP-based methods for constructing CPQ Lyapunov functions [41], is of interest. Finally, methods for synthesizing controllers with LP for hybrid systems can be investigated, where we could leverage the presented methods for calculating upper bounds on the L_1-gain and H_1-norm for controller optimization.

CRediT authorship contribution statement

Juan Javier Palacios Roman: Conceptualization, Formal analysis, Investigation, Software, Writing - original draft. **Sigurdur F. Hafstein:** Conceptualization, Supervision, Writing - review and editing. **Sebastiaan J.A.M. van den Eijnden:** Conceptualization, Supervision, Writing - review and editing. **Marcel Heertjes:** Conceptualization, Supervision, Writing - review and editing. **W.P.M.H. Heemels:** Conceptualization, Supervision, Writing - review and editing, Funding acquisition.

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