Constructing Piecewise Quadratic Lyapunov Functions with Linear Programming for Continuous-Time Switched and Conewise Linear Systems ⋆

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Abstract

In this paper, we present a method for constructing continuous piecewise quadratic (CPQ) Lyapunov functions for continuous-time switched and conewise linear systems using linear programming (LP). Key in our approach is the formulation of effective sufficient conditions for the copositivity of matrices via diagonal dominance. This formulation consists of linear constraints and can be expressed as an LP. It is shown that the sufficient conditions are also necessary conditions for the existence of a Lyapunov function, given a sufficiently refined CPQ function. We provide an in-depth comparison between our new method and other computational methods in the literature, and provide extensive numerical experiments on various switched and conewise linear systems. In particular, we show that the proposed method is the most accurate of the LP-based methods for constructing Lyapunov functions and is a numerically competitive alternative to LMI-based methods.

Key words: Stability of hybrid systems, Lyapunov methods, Linear programming.

1 Introduction

Switched linear systems (SLSs) and *conewise linear systems* (CLSs) represent a particular class of dynamical systems formed by a collection of linear subsystems that are combined with time and/or state-based switching rules [30]. In recent years, SLSs and CLSs have become increasingly important within a wide range of applications, including power converters [34], traffic management [33], epidemiological modeling [2] and high-performance motion control systems [43, 41]. Assessing stability of SLSs and CLSs is a well-studied problem, and over the years many constructive tools for stability analysis have been presented, see, e.g., [31] for an overview. Examples range from dwell-time approaches [48, 47], to frequency-domain techniques [28, 42] and Lyapunov-based methods [9, 26, 40, 27].

Of particular interest is the use of *continuous piecewise quadratic* (CPQ) Lyapunov functions [26] for the stability analysis of continuous-time SLSs and CLSs. The specific interest in CPQ Lyapunov functions stems from the possibility to express stability tests in terms of convex optimization problems that can be solved efficiently [8, 39], and from their success in reducing conservatism compared to earlier methods in the analysis of CLSs, see, e.g., the seminal works in [26, 39]. The key idea underlying CPQ methods is to partition (part of) the state-space into smaller regions, to each of which a quadratic function is assigned that is continuous over the partition boundaries, and satisfies certain conditions locally. These conditions are typically expressed in terms of constrained inequalities that can be transformed into linear matrix inequalities (LMIs) by means of the so-called S-procedure [46]. For a partitioning of the state space with sufficiently small regions, these LMIs must admit a solution for an exponentially stable CLS. An alternative method that does not make explicit use of the S-procedure has been proposed in [23, 24], where stability

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conditions are expressed in terms of a copositive problem, i.e., the problem of verifying whether a certain matrix is copositive or, equivalently, has a corresponding quadratic form that is positive definite on the positive orthant [18]. This problem, in turn, is translated into an LMI problem.

The aforementioned results hinge on the idea of reformulating constrained inequalities as LMIs. Although LMI conditions facilitate numerically tractable stability tests for SLSs and CLSs, they scale badly in terms of computational complexity with increasing system dimensions and the number of state-space partitions. These qualities motivate the development of alternative methods for constructing CPQ Lyapunov functions that preserve the structure and numerical tractability coming from LMI-based methods, but require less computational effort (even in case of a large number of regimes in the state-space partitioning).

Therefore, we present in this paper a new approach for constructing CPQ Lyapunov functions for SLSs with arbitrary switching and CLSs using linear programming (LP). Compared to solving large-scale LMI problems using semidefinite programming (SDP), solving large LP problems can be done more efficiently and reliably as a result of their significantly simpler structure, see, e.g., [44, 8] for further discussions. The use of LP for stability analysis of SLSs and CLSs has been proposed before: For example, LP can be used to construct continuous piecewise affine (CPA) Lyapunov functions [19, 3, 20]. However, it was shown in [37] on a numerical example that CPA Lyapunov functions were less effective than CPQ Lyapunov functions, despite the computational benefit of LP over SDP, because CPA Lyapunov functions required a much larger number of piecewise segments. Moreover, in [10, 22], sufficient linear inequalities were used to prove the existence of a CPQ Lyapunov function by solving a copositive problem. However, these conditions are more conservative than the conditions we will present, which will be discussed in Section 6.3.

In line with the above, the main contributions of this paper are as follows. We present efficient conditions for the computation of CPQ Lyapunov functions for continuous-time SLSs and CLSs based on LP. Key in our approach is the reformulation of copositive matrix problems into (sufficient) tests that require verifying diagonal dominance of the involved matrices. These tests are subject to linear constraints that can be formulated as an LP problem. Our results apply to both SLSs with arbitrary switching and CLSs. For both of these classes we will show that for a sufficiently fine partitioning of the state space our tests become necessary conditions for exponential stability. We compare our method to other stability methods in the literature, including the use of CPQ and CPA Lyapunov functions, and provide extensive numerical experiments showcasing our method in comparison to other methods in the literature. In particular, we show theoretically and numerically that the proposed method is the least conservative of the LP-based methods, among [19] and [22], given a fixed partitioning of the state-space and numerically that the proposed method is computationally competitive with the LMI-based method in [26].

The remainder of this paper is structured as follows. We provide the necessary preliminaries in Section 2 such as the parametrization of CPQ Lyapunov functions. Section 3 will specify the systems under consideration and the stability analysis problem to be solved. In Section 4 we present our main results in the form of a method that exploits LP for computing CPQ Lyapunov functions for SLSs with arbitrary switching. In Section 5 we present similar results for CLSs. In Section 6 we provide an in-depth comparison of our LP-based method and other stability methods found in the literature, including computations of CPA functions through LP and CPQ functions through LMIs. Section 7 provides extensive numerical case studies on various systems demonstrating the benefits of our approach. We close the paper with conclusions and directions for future work in Section 8. Finally, Appendix A contains supporting theorems.

2 Preliminaries

In this paper, the subset of $\mathbb{R}$ with nonnegative value is denoted by $\mathbb{R}_{\geqslant 0}$. By $[A]_{ij}$ we denote the i,j-th entry of the matrix $A \in \mathbb{R}^{n\times n}$, $i,j \in \{1,2,\ldots,n\}$. We say that a symmetric matrix $A = A^\top \in \mathbb{R}^{n\times n}$ is positive definite (negative definite), denoted by $A \succ 0$ ($A \prec 0$), if $x^\top Ax > 0$ ($x^\top Ax < 0$) for all $x \in \mathbb{R}^n \backslash \{\underline{0}\}$, where $\underline{0}$ denotes the origin of $\mathbb{R}^n$. Moreover, we say that a symmetric matrix $A = A^\top \in \mathbb{R}^{n\times n}$ is positive definite (negative definite) on the set $\mathcal{B} \subseteq \mathbb{R}^n$, denoted by $A \succ_{\mathcal{B}} 0$ ($A \prec_{\mathcal{B}} 0$), if $x^\top Ax > 0$ ($x^\top Ax < 0$) for all $x \in \mathcal{B}\backslash\{\underline{0}\}$. If $A \succ_{\mathcal{B}} 0$ and $\mathcal{B} = \mathbb{R}^n_{\geqslant 0}$, then A is said to be a copositive matrix. Finally, a matrix $A \in \mathbb{R}^{n\times n}$ is said to be strictly diagonally dominant if $2 \cdot |[A]_{ii}| > \sum_{j=1}^n \big|[A]_{ij}\big|$ for all $i \in \{1,2,\ldots,n\}$.

Inequalities for vectors hold componentwise. Furthermore, $\|x\| := \sqrt{\sum_{i=1}^n x_i^2}$ and $\|x\|_\infty := \max_{i=1,2,\ldots,n} |x_i|$ denote the Euclidean norm and infinity norm of $x \in \mathbb{R}^n$, respectively, and $\|A\| := \max_{\|x\|=1} \|Ax\|$ denotes the matrix norm of $A \in \mathbb{R}^{n\times n}$ induced by the Euclidean norm.

2.1 Triangulations

A triangulation $\mathcal{T} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N\}$, $N \in \mathbb{N}$, is a collection of simplices $\mathcal{S}_i$, $i \in \{1,2,\ldots,N\}$, defined as

$$\mathcal{S}_i = \mathrm{co}(r_0^i, r_1^i, \ldots, r_n^i) := \left\{ \sum_{k=0}^n \lambda_k r_k^i \;\middle|\; \sum_{k=0}^n \lambda_k = 1, \lambda_k \geqslant 0, k \in \{0,1,\ldots,n\} \right\},$$

where $\mathrm{co}(\cdot)$ denotes the convex hull and $r_0^i, r_1^i, \ldots, r_n^i \in \mathbb{R}^n$ are affinely independent vertices. The affine independence of the vertices implies that each simplex $\mathcal{S}_i$, $i \in \{1, 2, \ldots, N\}$, has a non-empty interior. Furthermore, $\{v_0, v_1, \ldots, v_p\} \subset \mathbb{R}^n$, $p \in \mathbb{N}$, denotes the collection of all vertices of the simplices used in the triangulation, such that $\{r_0^i, r_1^i, \ldots, r_n^i\} \subset \{v_0, v_1, \ldots, v_p\}$ for all $i \in \{1, 2, \ldots, N\}$. Clearly, for all $i \in \{1, 2, \ldots, N\}$, there exists then an index function $g_i : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, p\}$ such that $r_k^i = v_{g_i(k)}$ for all $k \in \{0, 1, \ldots, n\}$.

We require that any two simplices $\mathcal{S}_i$ and $\mathcal{S}_j$, $i, j \in \{1, 2, \ldots, N\}$ and $i \neq j$, either have an empty intersection or the intersection is a polyhedron described by the convex hull of the common vertices of the simplices $\mathcal{S}_i$ and $\mathcal{S}_j$. We further require that $\bigcup_{i=1}^N \mathcal{S}_i$ is a neighbourhood of the origin and that $r_0^i = v_0 = \underline{0}$ for all $i \in \{1, 2, \ldots, N\}$. Then $r_1^i, r_2^i, \ldots, r_n^i$ are linearly independent for all $i \in \{1, 2, \ldots, N\}$. Examples of triangulations that satisfy these conditions are the triangular fans $K^{-1}\mathcal{T}_K$ and $\mathcal{T}_K$ from [20]. Here, $K \in \mathbb{N}$ is a measure of how refined the triangulation is, namely for all simplices $\mathcal{S}_i$, $i \in \{1, 2, \ldots, N\}$, in $K^{-1}\mathcal{T}_K$ we have that $\left\| r_k^i - r_\ell^i \right\|_2 \leqslant \sqrt{n}/K$ for all $k, \ell \in \{1, 2, \ldots, n\}$. Furthermore, $N = 2^n \cdot K^{n-1} \cdot n!$ and $\{v_1, v_2, \ldots, v_p\} = \{K^{-1}z \mid z \in \mathbb{Z}^n, \|z\|_\infty = K\}$. The triangulation $\mathcal{T}_K$ is a scaled version of $K^{-1}\mathcal{T}_K$ which we will use in Section 7. For further details, see [20].

2.2 Parameterization of CPQ functions

Given a triangulation $\mathcal{T}$ satisfying all conditions and assumptions above, we can associate a simplicial cone $\mathcal{C}_i$ to each simplex $\mathcal{S}_i$, $i \in \{1, 2, \ldots, N\}$, defined as

$$\begin{aligned} \mathcal{C}_i &= \mathrm{pos}(r_1^i, r_2^i, \ldots, r_n^i) \\ &:= \left\{ \sum_{k=1}^n \lambda_k r_k^i \;\middle|\; \lambda_k \geqslant 0, k \in \{1, 2, \ldots, n\} \right\}, \end{aligned} \tag{1}$$

where $\mathrm{pos}(\cdot)$ denotes the positive hull. Since $\mathcal{T}$ is a triangulation of a neighbourhood of the origin, the union of all $\mathcal{C}_i$ is equal to $\mathbb{R}^n$. Note that the triangulations $\mathcal{T}_K$ and $K^{-1}\mathcal{T}_K$ from [20] give the same conical subdivision of $\mathbb{R}^n$. An example of such a conical subdivision is given in Fig. 1.

Then, a CPQ function $V : \mathbb{R}^n \to \mathbb{R}$ can be defined as

$$V(x) = x^\top P_i x \qquad \text{if } x \in \mathcal{C}_i. \tag{2}$$

The matrices $P_i = P_i^\top \in \mathbb{R}^{n \times n}$ are parameterized by a matrix $\Phi = \Phi^\top \in \mathbb{R}^{p \times p}$ as follows:

$$\begin{aligned} [\Psi_i]_{k\ell} &= [\Phi]_{g_i(k) g_i(\ell)} && \forall k, \ell \in \{1, 2, \ldots, n\} \\ R_i &= \begin{bmatrix} r_1^i & r_2^i & \ldots & r_n^i \end{bmatrix} && \in \mathbb{R}^{n \times n} \\ E_i &= R_i^{-1} && \in \mathbb{R}^{n \times n} \\ P_i &= E_i^\top \Psi_i E_i && \in \mathbb{R}^{n \times n}. \end{aligned} \tag{3}$$

Note that vertices $r_1^i, r_2^i, \ldots, r_n^i$ are linearly independent and thus R_i is invertible and we assume that the order of the vertices $r_1^i, r_2^i, \ldots, r_n^i$ is fixed, in accordance with the definition of g_i in Section 2.1, so that R_i is uniquely defined. Note that if $x \in \mathcal{C}_i$, then $x = R_i \lambda$, where $\lambda := \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \end{bmatrix}^\top \in \mathbb{R}^n_{\geqslant 0}$ is the vector with the λ_k in (1) as its entries, and thus

$$\begin{aligned} V(x) &= x^\top P_i x = \lambda^\top R_i^\top P_i R_i \lambda \\ &= \lambda^\top R_i^\top E_i^\top \Psi_i E_i R_i \lambda = \lambda^\top \Psi_i \lambda \\ &= \sum_{k=1}^n \sum_{\ell=1}^n \lambda_k \lambda_\ell [\Psi_i]_{k\ell} = \sum_{k=1}^n \sum_{\ell=1}^n \lambda_k \lambda_\ell [\Phi]_{g_i(k) g_i(\ell)}. \end{aligned} \tag{4}$$

Remark 1 *This method of parameterizing V was suggested in [39]. It can be interpreted as follows: The value of $V(v_m) = [\Phi]_{mm}$ for all $m \in \{1, 2, \ldots, p\}$. Moreover, $V(r_k^i) = [\Phi]_{g_i(k) g_i(k)}$ and*

$$\begin{aligned} V\left(\frac{1}{2} r_k^i + \frac{1}{2} r_\ell^i\right) =& \frac{1}{4} [\Phi]_{g_i(k) g_i(k)} + \frac{1}{4} [\Phi]_{g_i(k) g_i(\ell)} \\ &+ \frac{1}{4} [\Phi]_{g_i(\ell) g_i(k)} + \frac{1}{4} [\Phi]_{g_i(\ell) g_i(\ell)} \end{aligned} \tag{5}$$

for all $i \in \{1, 2, \ldots, N\}$ and $k, \ell \in \{1, 2, \ldots, n\}$.

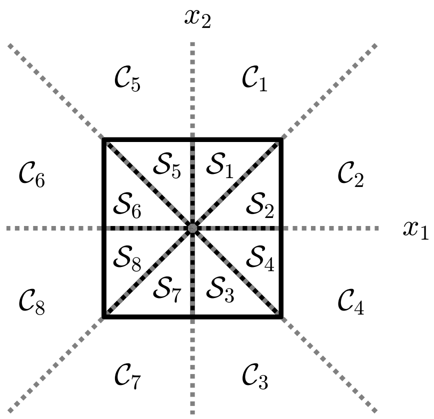

Fig. 1. Visualization of the conic subdivision $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_N\}$ that results from the triangulation $\mathcal{T}_1$ of a two-dimensional space (x_1, x_2). The boundaries of the simplices $\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N\}$ of the triangulation $\mathcal{T}_1$ are indicated with black lines and the boundaries of the cones $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_N\}$ with gray dotted lines.

By construction, V will be well-defined and continuous. This is obvious for the case when x is in the interior of $\mathcal{C}_i$. In case $x \in \mathcal{C}_i \cap \mathcal{C}_j$, where $i, j \in \{1, 2, \dots, N\}$ and $i \neq j$, then x is a linear combination of the common (extreme) rays of the cones $\mathcal{C}_i$ and $\mathcal{C}_j$, denoted by $\{y_1, y_2, \dots, y_m\} := \{r_1^i, r_2^i, \dots, r_n^i\} \cap \{r_1^j, r_2^j, \dots, r_n^j\} \subset \{v_1, v_2, \dots, v_p\}$. In other words,

$$x = \sum_{k=1}^{m} \lambda_k y_k = \sum_{k=1}^{m} \lambda_k r_{s_k}^i = \sum_{k=1}^{m} \lambda_k r_{t_k}^j$$

where $0 < m < n$ and $y_k = r_{s_k}^i = r_{t_k}^j$ for some $s_k, t_k \in \{1, 2, \dots, n\}$ for all $k \in \{1, 2, \dots, m\}$. Thus, $g_i(s_k) = g_j(t_k)$ and by using (4) we can show that for all $x \in \mathcal{C}_i \cap \mathcal{C}_j$

$$\begin{aligned} V(x) = x^\top P_i x &= \sum_{k=1}^{m} \sum_{\ell=1}^{m} \lambda_k \lambda_\ell [\Phi]_{g_i(s_k) g_i(s_\ell)} \\ &= \sum_{k=1}^{m} \sum_{\ell=1}^{m} \lambda_k \lambda_\ell [\Phi]_{g_j(t_k) g_j(t_\ell)} = x^\top P_j x, \end{aligned}$$

which shows that V is continuous and well-defined.

3 System classes and problem statement

In this section we will introduce the classes of systems under consideration and relevant stability theorems. Finally, the main problem statement will be specified.

3.1 Switched linear systems with arbitrary switching

Consider the continuous-time SLS

$$\dot{x}(t) = A_{\sigma(t)} x(t), \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the state at time $t \in \mathbb{R}_{\geqslant 0}$, $\sigma : \mathbb{R}_{\geqslant 0} \to \{1, 2, \dots, M\}$ is a piecewise constant function that is right-continuous and only has a finite number of discontinuity points on every finite time-interval, called the switching signal and $A_j \in \mathbb{R}^{n \times n}$ for all $j \in \{1, 2, \dots, M\}$. A solution $x\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}^n$ to (6) is understood in the sense of Carathéodory [45].

We use the following definition and theorem adapted from [30] and [21].

Definition 2 *The SLS* (6) *is said to be globally uniformly exponentially stable (GUES), if there exist constants* $c, \lambda > 0$ *such that for all switching signals* σ*, i.e., under arbitrary switching, all solutions* x *of* (6) *satisfy*

$$\|x(t)\| \leqslant c \|x(0)\| \, e^{-\lambda t}$$

for all $t \in \mathbb{R}_{\geqslant 0}$.

Theorem 3 *The SLS* (6) *is GUES, if there exists a common CPQ Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}$ *as in* (2) *that satisfies for all* $i \in \{1, 2, \dots, N\}$ *and* $j \in \{1, 2, \dots, M\}$

$$c_1 \|x\|^2 \leqslant x^\top P_i x \leqslant c_2 \|x\|^2 \qquad \forall x \in \mathcal{C}_i \tag{7a}$$
$$x^\top (A_j^\top P_i + P_i A_j) x \leqslant -c_3 \|x\|^2 \qquad \forall x \in \mathcal{C}_i \tag{7b}$$

for some $c_1, c_2, c_3 > 0$.

Note that the cones $\mathcal{C}_i$ are not related to the system modes A_j. Instead, the cones $\mathcal{C}_i$ are a construct for building the CPQ Lyapunov function V only, which must hold for any active system mode across the entire state-space.

3.2 Conewise linear systems

Consider the continuous-time CLS [11]

$$\dot{x}(t) = A_j x(t) \qquad \text{if } x(t) \in \mathcal{F}_j, \tag{8}$$

where $x(t) \in \mathbb{R}^n$ is the state at time $t \in \mathbb{R}_{\geqslant 0}$ and $A_j \in \mathbb{R}^{n \times n}$ and $\mathcal{F}_j \subset \mathbb{R}^n$ for all $j \in \{1, 2, \dots, M\}$. We assume that the union of the subsets $\mathcal{F}_j$ is equal to $\mathbb{R}^n$, the intersection of the interiors of $\mathcal{F}_i$ and $\mathcal{F}_j$ is empty for all $i, j \in \{1, 2, \dots, M\}$, $i \neq j$, and the closure of $\mathcal{F}_j$ is a simplicial cone for all $j \in \{1, 2, \dots, M\}$. These systems are also referred to as switched linear systems with state-dependent switching and are a subclass of piecewise linear systems. Solutions to (8) are understood in the sense of Filippov, i.e., are solutions to the differential inclusion

$$\dot{x}(t) \in F(x(t)) := \operatorname{co} \left\{ \bigcup_{j=1}^{M} \{A_j x(t) \mid x(t) \in \bar{\mathcal{F}}_j\} \right\}, \tag{9}$$

where $\bar{\mathcal{F}}_j$ is the closure of the set $\mathcal{F}_j$. Thus, potential sliding modes are taken into account, see, e.g., [6, 14, 13]. An example of a CLS is given in Fig. 2.

We use the following definition and theorem adapted from [21] and [29].

Definition 4 *The CLS* (8) *is said to be globally exponentially stable (GES), if there exist constants* $c, \lambda > 0$ *such that for all Filippov solutions* x *to* (8)*, i.e., for all solutions to* (9)*, it holds that*

$$\|x(t)\| \leqslant c \|x(0)\| \, e^{-\lambda t}$$

for all $t \in \mathbb{R}_{\geqslant 0}$.

Theorem 5 *The CLS* (8) *is GES, if there exists a CPQ Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}$ *as in* (2) *that satisfies for all* $i \in \{1, 2, \dots, N\}$ *and* $j \in \{1, 2, \dots, M\}$

$$c_1 \|x\|^2 \leqslant x^\top P_i x \leqslant c_2 \|x\|^2 \qquad \forall x \in \mathcal{C}_i \tag{10a}$$
$$x^\top (A_j^\top P_i + P_i A_j) x \leqslant -c_3 \|x\|^2 \quad \forall x \in \mathcal{C}_i \cap \bar{\mathcal{F}}_j, \tag{10b}$$

for some $c_1, c_2, c_3 > 0$.

3.3 Problem definition

Our goal is to find a CPQ function V that satisfies condition (7b) for an SLS or conditions (10b) for a CLS, i.e., a CPQ function V such that $V(x(t))$ is decreasing over time along all solutions x of the system, except for the zero-solution. Then:

(i) if the resulting function V additionally satisfies (7a) or (10a), i.e., if V is positive definite, then the SLS or CLS is GUES or GES, respectively;
(ii) if the resulting function V does not satisfy (7a) or (10a), i.e., if V is not positive definite, then the SLS or CLS is unstable.

The first statement follows from Theorem 3 or 5. The second statement follows from Lemma 4.1 in [25] and a sketch of proof is provided here for completeness: Suppose there exists a CPQ function V that, given a certain SLS, satisfies (7b), but does not satisfy (7a). Then, there exists a state $\hat{x}(0) \neq \underline{0}$ such that $V(\hat{x}(0)) \leqslant 0$. Consider the solution $\hat{x}(t)$ that starts at $\hat{x}(0)$. Because V satisfies (7b), V is decreasing over time along all solutions of the system, except for the zero-solution. Thus, after a time step δ, it will hold that $V(\hat{x}(\delta)) < 0$. In fact, as $t \to \infty$, $V(\hat{x}(t)) \to -\infty$, which means $||\hat{x}(t)|| \to \infty$ as $t \to \infty$. This implies instability of the SLS. A similar argument can be made for a CLS.

This method is known as the two-step procedure [25, Proposition 4.4], where the first step is finding a function V that is decreasing over time along all solutions of the system except for the zero solution, and the second step

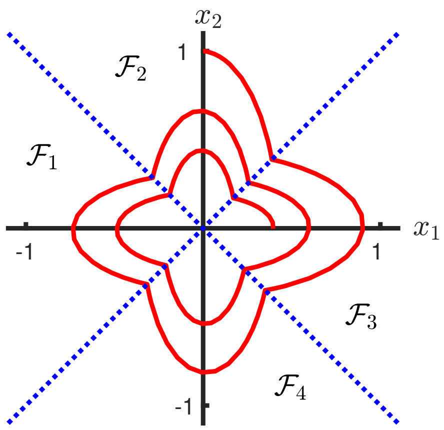

Fig. 2. Example of a CLS taken from [25], where $M = 4$ and $n = 2$. The definition of the system is given in (47)-(49). The boundaries of the sets $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ are indicated with blue dotted lines and a sample trajectory starting at $(x_1, x_2) = (0, 1)$ with red. When the state enters a new set, the differential equations governing the dynamics change.

is checking whether V is positive definite. This reduces the size of an optimization problem to find a suitable V, since less constraints have to be taken into account.

4 CPQ Lyapunov functions for SLSs

In this section we will show sufficient conditions in the form of linear inequalities under which a CPQ function is a Lyapunov function for an SLS (6). Then, a converse theorem will be presented that shows that for a sufficiently refined CPQ function the conditions become necessary, when the system is GUES.

4.1 Sufficient conditions for CPQ Lyapunov functions of SLSs

The CPQ function V is a Lyapunov function to the SLS (6), i.e., satisfies (7a) and (7b), if

$$P_i \succ_{\mathcal{C}_i} 0 \tag{11a}$$
$$Q_{ij} \prec_{\mathcal{C}_i} 0 \tag{11b}$$

for all $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, M\}$, where

$$Q_{ij} := A_j^\top P_i + P_i A_j. \tag{12}$$

For the first step of the two-step procedure in Section 3.3, a CPQ function V that satisfies (11b) needs to be constructed. We wish to do this with an LP problem. Thereto, consider the following theorem.

Theorem 6 *Consider a CPQ function V, parameterized as in (2), and the SLS (6). Define, for all $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, M\}$, the matrices*

$$B_{ij} = R_i^\top Q_{ij} R_i, \tag{13}$$

with Q_{ij} as in (12), and $C_{ij} \in \mathbb{R}^{n \times n}$ such that

$$[C_{ij}]_{kk} = [B_{ij}]_{kk} \qquad \forall k \in \{1, 2, \ldots, n\} \tag{14}$$
$$[C_{ij}]_{k\ell} = \max([B_{ij}]_{k\ell}, 0) \quad \forall k, \ell \in \{1, 2, \ldots, n\}, k \neq \ell. \tag{15}$$

If for all $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, M\}$, the sum of each row of C_{ij} is negative, i.e.,

$$\sum_{\ell=1}^{n} [C_{ij}]_{k\ell} < 0 \quad \forall k \in \{1, 2, \ldots, n\}, \tag{16}$$

then V satisfies condition (11b).

PROOF. First, we prove that (15) and (16) imply that C_{ij} is a negative definite matrix, i.e., a matrix with all eigenvalues strictly negative. Note that C_{ij} is a real symmetric matrix, and therefore its eigenvalues are real. Then, Gerschgorin's circle theorem [15] states that all eigenvalues of C_{ij} must lie in the set

$$\bigcup_{k\in\{1,2,\ldots,n\}} \{a \in \mathbb{R} \mid [C_{ij}]_{kk} - \sum_{\substack{\ell=1\\ \ell\neq k}}^{n} \big|[C_{ij}]_{k\ell}\big| \leqslant a \leqslant [C_{ij}]_{kk} + \sum_{\substack{\ell=1\\ \ell\neq k}}^{n} \big|[C_{ij}]_{k\ell}\big|\}. \quad (17)$$

Due to (15), the off-diagonal entries of C_{ij} are all nonnegative. Therefore,

$$[C_{ij}]_{kk} + \sum_{\substack{\ell=1\\ \ell\neq k}}^{n} \big|[C_{ij}]_{k\ell}\big| = \sum_{\ell=1}^{n} [C_{ij}]_{k\ell}.$$

Thus, (16) implies that the set (17) only contains negative numbers, which means C_{ij} must be a negative definite matrix.

Secondly, we show that $\lambda^\top B_{ij}\lambda \leqslant \lambda^\top C_{ij}\lambda < 0$ for all $\lambda \in \mathbb{R}^n_{\geqslant 0}\backslash\{\underline{0}\}$. The second inequality holds because C_{ij} is a negative definite matrix. The first inequality follows from

$$\begin{aligned}
\lambda^\top C_{ij}\lambda &= \sum_{k=1}^{n} \lambda_k^2 [C_{ij}]_{kk} + \sum_{k=1}^{n} \sum_{\substack{\ell=1\\ \ell\neq k}}^{n} \lambda_k\lambda_\ell [C_{ij}]_{k\ell} \\
&= \sum_{k=1}^{n} \lambda_k^2 [B_{ij}]_{kk} + \sum_{k=1}^{n} \sum_{\substack{\ell=1\\ \ell\neq k}}^{n} \lambda_k\lambda_\ell \max([B_{ij}]_{k\ell}, 0) \\
&\geqslant \sum_{k=1}^{n} \lambda_k^2 [B_{ij}]_{kk} + \sum_{k=1}^{n} \sum_{\substack{\ell=1\\ \ell\neq k}}^{n} \lambda_k\lambda_\ell [B_{ij}]_{k\ell} \\
&= \lambda^\top B_{ij}\lambda.
\end{aligned}$$

Finally, we show that $\lambda^\top B_{ij}\lambda < 0$ for all $\lambda \in \mathbb{R}^n_{\geqslant 0}\backslash\{\underline{0}\}$ implies that $Q_{ij} \prec_{\mathcal{C}_i} 0$. Note that (1) can equivalently be expressed as

$$\mathcal{C}_i = \left\{ x \in \mathbb{R}^n \; | E_i x = \lambda \in \mathbb{R}^n_{\geqslant 0} \right\},$$

where E_i is defined in (3). Then, if we substitute λ with $E_i x$, B_{ij} with (13) and R_i with E_i^{-1}, we obtain

$$\begin{aligned}
& \lambda^\top B_{ij}\lambda < 0 \quad \forall \lambda \in \mathbb{R}^n_{\geqslant 0}\backslash\{\underline{0}\} \\
\Longrightarrow\quad & x^\top E_i^\top (E_i^{-1})^\top Q_{ij} E_i^{-1} E_i x < 0 \quad \forall E_i x \in \mathbb{R}^n_{\geqslant 0}\backslash\{\underline{0}\} \\
\Longrightarrow\quad & x^\top Q_{ij} x < 0 \quad \forall x \in \mathcal{C}_i\backslash\{\underline{0}\},
\end{aligned}$$

which completes the proof. □

The theorem states conditions under which Φ parameterizes a common CPQ Lyapunov function candidate V that satisfies (11b). Such a matrix Φ can be found by solving the following feasibility problem:

$$\begin{aligned}
&\text{find } \Phi, C_{ij} && (18)\\
&\text{s.t. } P_i = E_i^\top [\Phi]_{g_i(k) g_i(\ell)} E_i && \forall i,k,\ell\\
&\quad B_{ij} = R_i^\top (A_j^\top P_i + P_i A_j) R_i && \forall i,j\\
&\quad [C_{ij}]_{kk} = [B_{ij}]_{kk} && \forall i,j,k\\
&\quad [C_{ij}]_{k\ell} \geqslant [B_{ij}]_{k\ell} && \forall i,j,k,\ell, k\neq \ell \quad (19)\\
&\quad [C_{ij}]_{k\ell} \geqslant 0 && \forall i,j,k,\ell, k\neq \ell \quad (20)\\
&\quad \sum_{\ell=1}^{n} [C_{ij}]_{k\ell} < 0 && \forall i,j,k,
\end{aligned}$$

where E_i, g_i and R_i depend on the triangulation $\mathcal{T}$. Note that the nonlinear constraints (15) were replaced by the linear constraints (19) and (20) and that the matrices C_{ij} are being treated as slack variables in (18). This is so that (18) becomes an LP feasibility problem.

Remark 7 *The second step of the two-step procedure in Section 3.3, i.e., the verification that V satisfies* (11a)*, can be done with some conservatism by checking whether there exist $U_i = U_i^\top \in \mathbb{R}^{n\times n}_{\geqslant 0}$ such that*

$$P_i - E_i^\top U_i E_i \succ 0 \qquad \forall i \in \{1,2,\ldots,N\}, \qquad (21)$$

where the S-procedure [46] is used to validate (11a)*. This condition can be checked with an LMI solver. The second step of the two-step procedure can also be completed by verifying that*

$$[R_i^\top P_i R_i]_{k\ell} > 0 \qquad \forall i,k,\ell, \qquad (22)$$

since then

$$\begin{aligned}
V(x) &= x^\top P_i x && \forall x \in \mathcal{C}_i\\
&= \lambda^\top R_i^\top P_i R_i \lambda && \forall \lambda \in \mathbb{R}^n_{\geqslant 0}\\
&= \sum_{k=1}^{n}\sum_{\ell=1}^{n} \lambda_k\lambda_\ell [R_i^\top P_i R_i]_{k\ell} \geqslant 0 && \forall \lambda \in \mathbb{R}^n_{\geqslant 0}.
\end{aligned}$$

Conditions (22) *are more conservative than conditions* (21)*. However, conditions* (22) *can be verified with an LP solver and could even be included in the LP problem* (18)*, if we wish to skip the two-step procedure and construct a Lyapunov function in one step.*

Remark 8 *To ensure that V satisfies* (11b)*, the matrices $-B_{ij}$ are forced to be strictly copositive, i.e., $-\lambda^\top B_{ij}\lambda > 0$ for all $\lambda \in \mathbb{R}^n_{\geqslant 0}\backslash\{\underline{0}\}$. The construction of and the conditions on the matrices C_{ij} are all for the* *purpose of asserting this. The use of strictly copositive matrices for the analysis of SLSs and CLSs has been suggested before, see, e.g., [38, 12, 10, 22, 23, 24].*

4.2 Converse theorem

To prove a converse theorem, i.e., that our method always succeeds in finding a Lyapunov function to a GUES system (6), we largely follow the proof of [20, Thm. 1], where a similar converse theorem is proved for CPA Lyapunov functions for SLSs with arbitrary switching.

Theorem 9 *Assume that the system* (6) *is GUES. Then, when using the triangulation* $K^{-1}\mathcal{T}_K$, *the LP problem* (18) *is feasible for all large enough* $K \in \mathbb{N}$.

PROOF. To prove the feasibility of the LP problem (18), we assign values to all the variables and then show that the constraints of the problem are fulfilled when the variables have these values.

In [20, Lemma 2] it is shown that, if (6) is GUES, there exists a Lyapunov function V that:

- is infinitely differentiable, except at $x = \underline{0}$;
- is positively homogeneous of degree one;
- and satisfies for all x and some $c_1, c_2, c_3 > 0$

$$\begin{aligned} &c_1\|x\| \leqslant V(x) \leqslant c_2\|x\| \\ &\nabla V(x)A_jx \leqslant -c_3\|x\| \quad \forall j \in \{1,2,\ldots,M\}. \end{aligned}$$

If we define $W(x) = V(x)^2$, then W is a Lyapunov function for (6) as well that:

- is infinitely differentiable, except at $x = \underline{0}$;
- is positively homogeneous of degree two, i.e.,

$$W(cx) = c^2W(x) \quad \text{and} \quad W(x) = \frac{1}{2}\nabla W(x)x. \tag{23}$$

  for all $x \neq \underline{0}$ and $c > 0$. The second statement follows from Euler's homogeneous function theorem [5];
- and satisfies for all x and some $c_1, c_2, c_3 > 0$

$$\begin{aligned} &c_1\|x\|^2 \leqslant W(x) \leqslant c_2\|x\|^2 \\ &\nabla W(x)A_jx \leqslant -c_3\|x\|^2 \quad \forall j \in \{1,2,\ldots,M\}. \end{aligned}$$

We assign values to the variables of the LP problem by defining for every simplex $\mathcal{S}_i$ the symmetric matrix P_i through

$$(r_k^i + r_\ell^i)^\top P_i(r_k^i + r_\ell^i) = W(r_k^i + r_\ell^i) \tag{24}$$

for all $k, \ell \in \{1,2,\ldots,n\}$. That the resulting CPQ function (2) is still continuous and well-defined follows from [17, Thm. 2.8]. Define the functions $V_i\colon \mathbb{R}^n \to \mathbb{R}$ through $V_i(x) = x^\top P_i x$ and set, as before, $V(x) = V_i(x)$, if $x \in \mathcal{C}_i$. Then $V(r_k^i + r_\ell^i) = W(r_k^i + r_\ell^i)$ for all i, k, ℓ and we have from (24) that

$$2(r_k^i)^\top P_i r_\ell^i = W(r_k^i + r_\ell^i) - W(r_k^i) - W(r_\ell^i). \tag{25}$$

The values of the elements of Φ, i.e., the variables of the LP problem, are now obtained from the values of $V(r_k^i/2 + r_\ell^i/2)$ by using (5).

Define the constants

$$\bar{a} := \max_{j=1,2,\ldots,N} \|A_j\|, \tag{26}$$

$$\bar{g} := \max_{\|x\|_\infty=1} \|\nabla W(x)\|, \text{ and} \tag{27}$$

$$\bar{h} := \max_{\|x\|_\infty=1} \|H_W(x)\|, \tag{28}$$

where H_W is the Hessian matrix of W. We will show that all the entries of all the matrices B_{ij} in the LP problem (18) are negative, if

$$K > \frac{\sqrt{n}\bar{a}\left(\bar{g} + 9\sqrt{3}n^2\bar{h}\right)}{c_3}, \tag{29}$$

where $c_3 > 0$ is the constant from (7b). Then, a feasible solution is given by Φ and by setting $[C_{ij}]_{kk} = [B_{ij}]_{kk} < 0$ and $[C_{ij}]_{k\ell} = 0$, $k \neq \ell$, for all i, j, k, ℓ.

Let us first derive an upper bound on the difference between ∇W and ∇V_i at the vertices r_k^i. Let $\mathcal{S}_i \in K^{-1}\mathcal{T}_K$ be an arbitrary, but fixed simplex. Following the proof of [20, Thm. 1], we have

$$\|r_k^i\|_\infty = 1 \quad \text{and} \quad \|r_k^i - r_\ell^i\| \leqslant \frac{\sqrt{n}}{K} \tag{30}$$

for all $k, \ell \in \{1,2,\ldots,n\}$ and

$$\|E_i\| \leqslant K\sqrt{3(n+1)}. \tag{31}$$

Now, since $\nabla V_i(r_k^i) = 2(r_k^i)^\top P_i$ and $E_i = R_i^{-1}$, we have

$$\begin{aligned} \|\nabla V(r_k^i) - \nabla W(r_k^i)\| &= \|2(r_k^i)^\top P_i - \nabla W(r_k^i)\| \\ &\leqslant \|2(r_k^i)^\top P_i R_i - \nabla W(r_k^i)R_i\|\|E_i\|. \end{aligned} \tag{32}$$

Consider the ℓ-th component

$$y_\ell = 2(r_k^i)^\top P_i r_\ell^i - \nabla W(r_k^i) r_\ell^i \tag{33}$$

of the vector $y := 2(r_k^i)^\top P_i R_i - \nabla W(r_k^i)R_i$.

The case $\ell = k$ is simple: By (25) and (23) we have

$$y_\ell = 2W(r_\ell^i) - 2W(r_\ell^i) = 0 \quad \text{when } \ell = k.$$

To tackle the case $\ell \neq k$, consider first that by Taylor's theorem [5] we have

$$W(r_\ell^i) = W(r_k^i) + \nabla W(r_k^i)(r_\ell^i - r_k^i) + \frac{1}{2}(r_\ell^i - r_k^i)^\top H_W(x_{k,\ell})(r_\ell^i - r_k^i),$$

where $x_{k,\ell}$ is a point on the line segment between r_k^i and r_ℓ^i. Hence, there is a constant $\bar{h}_1$, $|\bar{h}_1| \leqslant \bar{h}$, such that

$$\begin{aligned}\nabla W(r_k^i)r_\ell^i &= W(r_\ell^i) - W(r_k^i) + \nabla W(r_k^i)r_k^i - \frac{n\bar{h}_1}{2K^2} \\ &= W(r_\ell^i) + W(r_k^i) - \frac{n\bar{h}_1}{2K^2}\end{aligned}$$

by (30) and (23). Hence, formula (33) can be written as

$$\begin{aligned}y_\ell &= W(r_k^i + r_\ell^i) - 2W(r_k^i) - 2W(r_\ell^i) + \frac{n\bar{h}_1}{2K^2} \\ &= 4\left[W\left(\frac{1}{2}r_k^i + \frac{1}{2}r_\ell^i\right) - \frac{1}{2}W(r_k^i) - \frac{1}{2}W(r_\ell^i)\right] + \frac{n\bar{h}_1}{2K^2}\end{aligned}$$

using (25) and (23). By [7, Prop. 4.1], (28) and (30) we get

$$\left|W\left(\frac{1}{2}r_k^i + \frac{1}{2}r_\ell^i\right) - \frac{1}{2}W(r_k^i) - \frac{1}{2}W(r_\ell^i)\right| \leqslant \frac{n\bar{h}}{K^2},$$

hence

$$|y_\ell| \leqslant \frac{4n\bar{h}}{K^2} + \frac{n|\bar{h}_1|}{2K^2} \leqslant \frac{9n\bar{h}}{2K^2} \quad \text{when } \ell \neq k.$$

Combining cases $\ell = k$ and $\ell \neq k$ delivers

$$\|y\| \leqslant \frac{9n\bar{h}\sqrt{n-1}}{2K^2}$$

and, by (32) and (31),

$$\|\nabla V(r_k^i) - \nabla W(r_k^i)\| \leqslant \|y\|\|E_i\| \leqslant \frac{9\sqrt{3}n^2\bar{h}}{2K}, \tag{34}$$

from which also

$$\|\nabla V(r_k^i)\| \leqslant \bar{g} + \frac{9\sqrt{3}n^2\bar{h}}{2K} \tag{35}$$

follows, see (27).

Consider the matrix $B_{ij} = R_i^\top(A_j^\top P_i + P_iA_j)R_i$ from the LP problem (40). Its k,ℓ-th element is given by

$$[B_{ij}]_{k\ell} = 2(r_k^i)^\top P_iA_jr_\ell^i = \nabla V_i(r_k^i)A_jr_\ell^i$$

and by (7b), (34), $\|x\|_\infty \leqslant \|x\| \leqslant \sqrt{n}\|x\|_\infty$, (35), and (30) we have

$$\begin{aligned}&\nabla V_i(r_k^i)A_jr_\ell^i \\ &\quad= \nabla W(r_k^i)A_jr_k^i + [\nabla V_i(r_k^i) - \nabla W(r_k^i)]A_jr_k^i \\ &\qquad+ \nabla V_i(r_k^i)A_j(r_\ell^i - r_k^i) \\ &\quad\leqslant -c_3\|r_k^i\|^2 \\ &\qquad\quad+ \|\nabla V_i(r_k^i) - \nabla W(r_k^i)\|\|A_j\|\|r_k^i\| \\ &\qquad\qquad+ \|\nabla V_i(r_k^i)\|\|A_j\|\|r_\ell^i - r_k^i\| \\ &\quad\leqslant -c_3 + \frac{9\sqrt{3}n^2\bar{h}}{2K} \cdot \bar{a} \cdot \sqrt{n} \\ &\qquad\quad+ \left(\bar{g} + \frac{9\sqrt{3}n^2\bar{h}}{2K}\right) \cdot \bar{a} \cdot \frac{\sqrt{n}}{K} \\ &\quad\leqslant -c_3 + \frac{\sqrt{n}\bar{a}}{K}\left(\bar{g} + 9\sqrt{3}n^2\bar{h}\right),\end{aligned}$$

which is negative for every K fulfilling (29). Hence, the LP problem is feasible and the proof is finished. □

Note that Theorem 9 follows a similar line of reasoning as the converse theorem in [20], which shows the existence of continuous piecewise affine (CPA) Lyapunov functions. It is important to note that the converse theorem presented here is however not a corollary to the converse theorem in [20]. Indeed, since the set of CPQ functions is disjoint from the set of CPA functions a separate converse theorem was needed. Moreover, since Theorem 6 uses linear constraints to enforce copositivity of matrices $-B_{ij}$, it needed to be verified whether these linear constraints can be satisfied for all GUES SLSs given a sufficiently refined triangulation $K^{-1}\mathcal{T}_K$, which is proven with Theorem 9.

5 CPQ Lyapunov functions for CLSs

Similar to the previous section, we will show sufficient conditions, that are linear in their arguments, under which a CPQ function is a Lyapunov function for a CLS (8). Then, a converse theorem will be presented that shows that for a sufficiently refined CPQ function the conditions become necessary for GES.

5.1 *Sufficient conditions for CPQ Lyapunov functions of CLSs*

To apply the method described in the previous section to the CLS (8), we must make the following assumption on the subsets $\mathcal{F}_j$ in (8) and $\mathcal{C}_i$ in (2). This assumption can always be satisfied by proper selection of the triangulation $\mathcal{T}$.

Assumption 10 *The intersection of* $\mathcal{C}_i$ *and the closure of* $\mathcal{F}_j$ *is given by*

$$\mathcal{C}_i \cap \bar{\mathcal{F}}_j = \mathrm{pos}(s_1^{ij}, s_2^{ij}, \ldots, s_{m_{ij}}^{ij}),$$

where $m_{ij} = 0$, $0 < m_{ij} < n$ *or* $m_{ij} = n$ *if* $\mathcal{C}_i \cap \bar{\mathcal{F}}_j$ *is* $\{\underline{0}\}$, *equal to a lower dimensional face of* $\mathcal{C}_i$ *or equal to* $\mathcal{C}_i$, *respectively, and* $\{s_1^{ij}, s_2^{ij}, \ldots, s_{m_{ij}}^{ij}\} \subseteq \{r_1^i, r_2^i, \ldots, r_n^i\}$, *for all* $i \in \{1, 2, \ldots, N\}$ *and* $j \in \{1, 2, \ldots, M\}$.

Assumption 10 can be interpreted as that the triangulation $\mathcal{T}$ must be chosen such that the closure of every set $\mathcal{F}_j$, $j \in \{1, 2, \ldots, M\}$, of the CLS (8) is equal to the union of some convex combinations of some vertices $v_0, v_1, \ldots v_p$ of the triangulation $\mathcal{T}$.

For brevity, let us define $\hat{\mathcal{C}}_{ij} := \mathcal{C}_i \cap \bar{\mathcal{F}}_j$ and

$$\hat{R}_{ij} := \begin{bmatrix} s_1^{ij} & s_2^{ij} & \ldots & s_{m_{ij}}^{ij} \end{bmatrix} \in \mathbb{R}^{n \times m_{ij}}. \tag{36}$$

The CPQ function V is a Lyapunov function to the CLS (8), i.e., satisfies (10a) and (10b), if

$$P_i \succ_{\mathcal{C}_i} 0 \tag{37a}$$
$$Q_{ij} \prec_{\hat{\mathcal{C}}_{ij}} 0 \tag{37b}$$

for all $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, ..., M\}$, where $\hat{\mathcal{C}}_{ij} \neq \{\underline{0}\}$, and with Q_{ij} as in (12). For the first step of the two-step procedure in Section 3.3, a CPQ function V that satisfies (37b) needs to be constructed. Hereto we can use a slightly modified version of Theorem 6.

Theorem 11 *Consider a CPQ function* V, *parameterized as in* (2), *the CLS* (8) *and a triangulation* $\mathcal{T}$ *that satisfies Assumption 10. Define, for all* $i \in \{1, 2, \ldots, N\}$ *and* $j \in \{1, 2, ..., M\}$ *where* $\hat{\mathcal{C}}_{ij} \neq \{\underline{0}\}$, *the matrices*

$$B_{ij} = \hat{R}_{ij}^\top Q_{ij} \hat{R}_{ij} \quad \in \mathbb{R}^{m_{ij} \times m_{ij}} \tag{38}$$

and $C_{ij} \in \mathbb{R}^{m_{ij} \times m_{ij}}$ *such that*

$$\begin{aligned} [C_{ij}]_{kk} &= [B_{ij}]_{kk} && \forall k \in \{1, 2, \ldots, m_{ij}\} \\ [C_{ij}]_{k\ell} &= \max([B_{ij}]_{k\ell}, 0) && \forall k, \ell \in \{1, 2, \ldots, m_{ij}\}, k \neq \ell. \end{aligned}$$

If for all $i \in \{1, 2, \ldots, N\}$ *and* $j \in \{1, 2, \ldots, M\}$ *where* $\hat{\mathcal{C}}_{ij} \neq \{\underline{0}\}$, *the sum of each row of* C_{ij} *is negative, i.e.,*

$$\sum_{\ell=1}^{m_{ij}} [C_{ij}]_{k\ell} < 0 \quad \forall k \in \{1, 2, \ldots, m_{ij}\},$$

then V *satisfies condition* (37b).

PROOF. The proof follows the same steps as the proof of Theorem 6, except in the final step, where it was shown that $\lambda^\top B_{ij} \lambda < 0$ for all $\lambda \in \mathbb{R}^n_{\geqslant 0} \backslash \{\underline{0}\}$ implies that $Q_{ij} \prec_{\mathcal{C}_i} 0$. Instead, it needs to be shown that

$$\lambda^\top B_{ij} \lambda < 0 \quad \forall \lambda \in \mathbb{R}^{m_{ij}}_{\geqslant 0} \backslash \{\underline{0}\} \implies Q_{ij} \prec_{\hat{\mathcal{C}}_{ij}} 0 \tag{39}$$

Note that $x \in \hat{\mathcal{C}}_{ij}$ is equivalent to $x = \hat{R}_{ij} \lambda$ for some $\lambda \in \mathbb{R}^{m_{ij}}_{\geqslant 0}$. Thus

$$x^\top Q_{ij} x < 0 \qquad \forall x \in \hat{\mathcal{C}}_{ij} \backslash \{\underline{0}\}$$

is equivalent to

$$\lambda^\top \hat{R}_{ij}^\top Q_{ij} \hat{R}_{ij} \lambda < 0 \qquad \forall \lambda \in \mathbb{R}^{m_{ij}}_{\geqslant 0} \backslash \{\underline{0}\}.$$

Since $B_{ij} = \hat{R}_{ij}^\top Q_{ij} \hat{R}_{ij}$, (39) has been proven and the proof is complete. □

Similar to the case of the SLS, an LP problem can be constructed such that a solution to the LP problem parameterizes a CPQ Lyapunov function candidate V that satisfies (37b):

$$\begin{aligned} &\text{find } \Phi, C_{ij} && \text{(40)} \\ &\text{s.t. } P_i = E_i^\top [\Phi]_{g_i(k) g_i(\ell)} E_i && \forall i, k, \ell \\ &\quad B_{ij} = R_i^\top (A_j^\top P_i + P_i A_j) R_i && \forall i, j, \hat{\mathcal{C}}_{ij} \neq \{\underline{0}\} \\ &\quad [C_{ij}]_{kk} = [B_{ij}]_{kk} && \forall i, j, k, \hat{\mathcal{C}}_{ij} \neq \{\underline{0}\} \\ &\quad [C_{ij}]_{k\ell} \geqslant [B_{ij}]_{k\ell} && \forall i, j, k, \ell, \hat{\mathcal{C}}_{ij} \neq \{\underline{0}\}, k \neq \ell \\ &\quad [C_{ij}]_{k\ell} \geqslant 0 && \forall i, j, k, \ell, \hat{\mathcal{C}}_{ij} \neq \{\underline{0}\}, k \neq \ell \\ &\quad \sum_{\ell=1}^{m_{ij}} [C_{ij}]_{k\ell} < 0 && \forall i, j, k, \hat{\mathcal{C}}_{ij} \neq \{\underline{0}\}. \end{aligned}$$

5.2 Converse theorem

Similar to the converse theorem in the previous section, we want to show that our method always succeeds in finding a Lyapunov function for a GES CLS (8).

Theorem 12 *Assume that system* (8) *is GES and that the triangulation* $K^{-1}\mathcal{T}_K$ *satisfies Assumption 10 for some* $K^\star \in \mathbb{N}$. *Then, the LP problem* (40) *is feasible for sufficiently large* $K \in \mathbb{N}$ *that satisfy* $K = 2^p K^\star$, *where* $p \in \mathbb{N}$.

PROOF. First, note that the condition $K = 2^p K^\star$ ensures that Assumption 10 is satisfied, which is a necessary condition to Theorem 11. Second, to prove the feasibility of the LP problem (40), we assign values to all the variables and then show that the constraints of the problem are fulfilled when the variables have these values by following the same steps as in the proof of Theorem 9.

Note that the systems in (8) and (9) are positively homogeneous of degree one, i.e., $F(cx(t)) = cF(x(t))$ for all $c > 0$, where $F(x(t))$ is the right-hand side of (9). If (8), or equivalently (9), is GES, it is shown in [36] that there exists a Lyapunov function W that:

- is infinitely differentiable;
- is positively homogeneous of degree two, i.e., satisfies (23);
- and satisfies for all x and some $c_1, c_2, c_3 > 0$

$$\begin{aligned} c_1\|x\|^2 \leqslant W(x) &\leqslant c_2\|x\|^2 \\ \nabla W(x) A_j x &\leqslant -c_3\|x\|^2 \quad \forall x \in \mathcal{F}_j, \\ & \qquad\qquad j \in \{1, 2, \ldots, M\}. \end{aligned}$$

Then, by following the same steps as in Theorem 9, we find that

$$\nabla V_i(r_k^i) A_j r_\ell^i \tag{41}$$

is negative for all $i \in \{1, 2, \ldots, N\}$, $j \in \{1, 2, \ldots, M\}$, $k, \ell \in \{1, 2, \ldots, n\}$ for which $r_k^i, r_\ell^i \in \mathcal{F}_j$, if

$$K > \frac{\sqrt{n}\bar{a}\left(\bar{g} + 9\sqrt{3}n^2\bar{h}\right)}{c_3},$$

where $\bar{a}$, $\bar{g}$ and $\bar{h}$ are defined in (26)-(28). If (41) is negative for all i, j, k, ℓ, then all entries of the matrices B_{ij} in (38) are negative and thus the LP problem (40) is feasible. □

Remark 13 *In the case that system* (8) *and the triangulation $K^{-1}\mathcal{T}_K$ do not satisfy Assumption 10 for any $K^\star \in \mathbb{N}$, a different triangulation $\mathcal{T}$ that does satisfy Assumption 10 could be used. The converse theorem only requires that the triangulation $\mathcal{T}$ satisfies the bounds in* (30) *and* (31). *This hints at a possible generalization of our converse theorems to triangulations other than the triangulation $K^{-1}\mathcal{T}_K$, which could potentially have numerical benefits.*

6 Comparison to similar methods

Alternative methods to assess the stability of SLSs and CLSs are:

- [26], which constructs CPQ Lyapunov functions using LMIs;
- [19], which constructs CPA Lyapunov functions using LP and;
- [22], which uses the results in [10] to construct CPQ Lyapunov functions using LP similarly to our method, but with different linear inequalities.

In this section we will highlight some differences between these methods and the proposed method.

6.1 Comparison to constructing CPQ Lyapunov functions with LMIs

If the method of [26] is applied to the SLS (6), then (11b) would be enforced by the linear matrix inequality

$$Q_{ij} + E_i^\top W_{ij} E_i \prec 0,$$

where $W_{ij} = W_{ij}^\top \in \mathbb{R}_{\geqslant 0}^{n \times n}$ would be a new optimization variable. This is equivalent to enforcing that $B_{ij} + W_{ij}$ is a negative definite matrix.

In our method, we enforce that C_{ij}, defined by (14) and (15), satisfies (16). It can be shown that this is equivalent to the existence of a matrix $W_{ij} = W_{ij}^\top \in \mathbb{R}_{\geqslant 0}^{n \times n}$ such that $C_{ij} := B_{ij} + W_{ij}$ is a strictly diagonally dominant and negative definite matrix, i.e.,

$$\left|[C_{ij}]_{kk}\right| > \sum_{\ell=1, \ell \neq k}^{n} \left|[C_{ij}]_{k\ell}\right| \tag{42}$$

and $[C_{ij}]_{kk} < 0$ for all $k \in \{1, 2, \ldots, n\}$. See Theorem 15 in Appendix A for the proof. The reason we do not use this equivalent formulation in the conditions of LP problems (18) and (40), is because the absolute operators in (42) would require a large number of slack variables, which would increase the number of optimization variables and linear constraints of the LP problems and therefore it is not computationally attractive.

Nevertheless, the alternative formulation shows that our method requires $B_{ij} + W_{ij}$ to be a negative definite and strictly diagonally dominant matrix, whereas the method in [26] solely requires it to be negative definite. Therefore, given the same triangulation $\mathcal{T}$, our method will always produce more conservative results. For example, the matrix

$$C_{ij} = B_{ij} + W_{ij} = \begin{bmatrix} -1 & 2 \\ 2 & -5 \end{bmatrix}$$

is negative definite, but not strictly diagonally dominant.

Our method, however, can benefit from *smaller* optimization problems and faster and more robust LP solvers [1]. Hence, there is a trade-off between accuracy and complexity. Note that both methods provide necessary and sufficient conditions for stability if the CPQ function uses a sufficiently fine partitioning of the state space [39]. Thus, the conservatism of our method compared to the LMI method goes away for a sufficiently refined partitioning, although the level of sufficient refinement for the LMI-based method will always be lower or equal to the level of sufficient refinement of our LP-based method.

6.2 Comparison to constructing CPA Lyapunov functions with LP

Linear constraints on $v_i^\top \in \mathbb{R}^n_{\geqslant 0}$ and $c_i \in \mathbb{R}$, $i \in \{1, 2, \ldots, N\}$, such that the CPA function

$$V(x) = v_i^\top E_i x + c_i \qquad \text{if } x \in \mathcal{C}_i,$$

is a Lyapunov function to a generic nonlinear system, were introduced in [19]. In [3] the method was applied to SLSs (6) and in [37] to CLSs.

A challenge with this method is that the number of piecewise segments N often needs to be much larger than when using CPQ functions. To illustrate this, consider the planar LTI system

$$\dot{x} = \begin{bmatrix} -\epsilon & -1 \\ 1 & -\epsilon \end{bmatrix} x, \tag{43}$$

where $\epsilon > 0$, as a special instance of the SLS (6), where $n = 2$ and $M = 1$. It was shown in [16, Lemma 3.2] that to construct a CPA Lyapunov function for (43), the Lyapunov function requires at least N piecewise segments such that

$$1 < \epsilon \sin\left(\frac{2\pi}{N}\right) + \cos\left(\frac{2\pi}{N}\right). \tag{44}$$

This means that for this specific instance of (6), the CPA method may require a computationally large LP problem to construct a CPA Lyapunov function. This behaviour can also become a computational limitation to the method when applying this method to other instances of SLSs (6) and CLSs (8), the latter of which was shown in [37].

By contrast, for any stable instance of (6) with $M = 1$, our method can always find a quadratic Lyapunov function, that satisfies the conditions of Theorem 6, provided that the triangulation $\mathcal{T}$ used to parameterize V satisfies the next assumption.

Assumption 14 *The vertices of a simplex $\mathcal{S}_i$ of $\mathcal{T}$ all lie in the same orthant, for all $i \in \{1, 2, \ldots, N\}$.*

The proof of this fact is given in Theorem 16 in Appendix A. This shows that for these specific instances of (6), our proposed method will require $N = 2^n$ simplices to construct a CPQ Lyapunov function with an LP problem, whereas the CPA method may require arbitrarily many simplices N, which is a computational advantage of the proposed method.

6.3 Comparison to constructing CPQ Lyapunov functions with different linear inequalities

In [22], linear inequalities are used to verify the existence of a CPQ Lyapunov function for stable piecewise linear differential inclusions. The linear inequalities are based on [10] and are similar to enforcing

$$\begin{aligned} -[B_{ij}]_{k\ell} > 0 \quad \forall i, j, k, \ell \\ [\hat{R}_{ij}^\top P_i \hat{R}_{ij}]_{k\ell} > 0 \quad \forall i, j, k, \ell, \end{aligned} \tag{45}$$

i.e., that $-B_{ij}$ and $\hat{R}_{ij}^\top P_i \hat{R}_{ij}$, where B_{ij}, $\hat{R}_{ij}$ and P_i are defined respectively in (38), (36) and (3), only have positive entries for all $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, M\}$. This enforces that $-B_{ij}$ and $\hat{R}_{ij}^\top P_i \hat{R}_{ij}$ are copositive matrices, which are sufficient and necessary conditions for a CPQ function to be a Lyapunov function.

Note however that a copositive matrix does not need to only have positive entries. For example, the matrix

$$-B_{ij} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

is copositive, but does not contain only positive entries. Whereas the linear inequalities in [10] and [22] would not be able to verify that $-B_{ij}$ is copositive, the proposed linear inequalities in (18) and (40) would, as the rows of B_{ij} sum to negative values, i.e., our inequalities are inherently less conservative. From numerical tests we found that for low dimensional systems ($n = 2$), the difference in conservativeness between the linear inequalities in [10] and [22] and our proposed linear inequalities is sufficiently small such that neither method is significantly computationally more beneficial than the other. However, for larger dimensional systems ($n > 2$), the difference in conservativeness increases, which can be explained by the growing number of off-diagonal elements in the matrices B_{ij} on which our linear inequalities pose less restrictive conditions. This causes our method to become computationally more efficient in comparison.

It is worth mentioning that in [10] and [22] their method was shown to always be able to verify whether or not a CPQ Lyapunov function could be constructed for a GUES SLS or GES CLS given a triangulation $\mathcal{T}$ within a finite number of steps. If no CPQ Lyapunov function could be constructed using the given triangulation $\mathcal{T}$, the cones of the triangulation $\mathcal{T}$ would have to be partitioned into smaller cones. However, no converse result was given that for a GUES SLS or GES CLS this procedure would eventually find a Lyapunov function. On the contrary, our converse Theorems 9 and 12 show that for a GUES SLS or GES CLS and given the triangulation $\mathcal{T}_K$, we will find a CPQ Lyapunov function that satisfies the conditions of Theorems 6 or 11, respectively, by stepwise increasing K.

7 Numerical case study

In this section, we will demonstrate the method presented in this paper for constructing CPQ Lyapunov functions with LP on elementary ($n = 2$) examples from the literature and on randomly generated SLSs of larger dimension ($n = 4$). We will compare the proposed CPQ (LP) method with the CPQ (LMI) method from [26], the CPA (LP) method from [19] and the CPQ (LP) method from [22].

To parameterize the Lyapunov function V, the triangular fan triangulation $\mathcal{T}_K$ mentioned in Section 2 is used. To solve the LP and LMI problems, the solver Mosek [35] implemented in MATLAB with Yalmip [32], is used. A description of an efficient implementation of our method in C++ is given in [4].

7.1 LTI system

Consider the normalized, planar LTI system (43) with $\epsilon = 0.01$. Using (44), a CPA Lyapunov function for the system would require a triangulation $\mathcal{T}$ with $N \geqslant 315$ piecewise segments. The triangulation $\mathcal{T}_{41}$ is used, since then $N = 2^n \cdot K^{n-1} \cdot n! = 328$. The CPQ (LMI) method comes down to solving the LMIs (11a) and (11b) where $N = M = 1$ and $\mathcal{C}_1 = \mathbb{R}^{n\times n}$. The CPQ (LP) method from [22] comes down to solving (45) with triangulation $\mathcal{T}_1$.

For the proposed CPQ (LP) method, we use the triangulation $\mathcal{T}_1$ ($N = 8$), which satisfies Assumption 14. By solving the LP problem (18) with additional linear constraints (22), a solution Φ, which parameterizes a CPQ Lyapunov function V, is found. See Fig. 3 for a visualization of V.

A comparison between the different methods is made in Table 1. A distinction is made between the total time

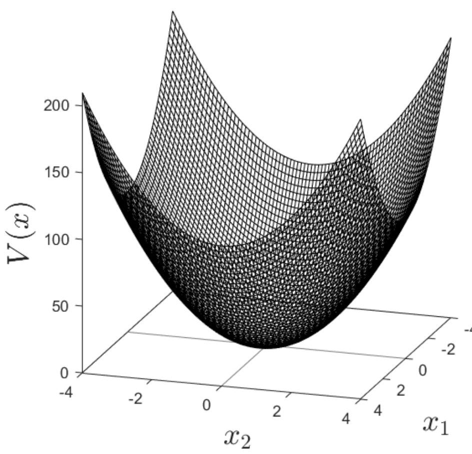

Fig. 3. CPQ Lyapunov function constructed with the proposed method for the LTI system (43) with $\epsilon = 0.01$ and $\mathcal{T}_1$.

Table 1
Comparison between methods for elementary examples.

	$N[-]$	$\tau_{\text{tot}}[\text{ms}]$	$\tau_{\text{sol}}[\text{ms}]$
LTI system (43)			
CPA (LP) [19]	328	467.7	4.893
CPQ (LMI) [26]	1	55.25	1.306
CPQ (LP) [22]	8	64.77	1.384
proposed CPQ (LP)	8	73.23	1.537
SLS (46)			
CPA (LP) [19]	400	1109	9.261
CPQ (LMI) [26]	32	226.2	10.18
CPQ (LP) [22]	56	161.9	3.707
proposed CPQ (LP)	32	186.1	3.711
CLS (47)-(49)			
CPA (LP) [19]	192	1181	3.603
CPQ (LMI) [26]	32	405.8	24.42
CPQ (LP) [22]	40	134.6	2.174
proposed CPQ (LP)	32	169.2	2.431

required to construct, solve and verify the solution of the LP/LMI problem, τ_{tot}, and the computation time required to solve the LP/LMI problem, τ_{sol}. While τ_{tot} can be reduced by using a more efficient implementation, such as the one presented in [4], τ_{sol} is more difficult to reduce and is therefore an important scalar when comparing the different methods.

It is clear that the CPA (LP) method uses the most computation time, which is due to the high number of simplices N required due to the low damping ϵ as discussed in Section 6.2. The CPQ (LMI) method is the fastest, which was expected as it only needs to solve two LMIs. The proposed CPQ (LP) method and the CPQ (LP) method from [22] are slightly slower than the CPQ (LMI) method, but the performance is comparable. For the two-dimensional examples, the CPQ (LP) methods will perform similarly: The method from [22] will be slightly faster due to less constraints and optimization variables in the LP problem, but may require a more refined triangulation, as mentioned in Section 6.3. In Section 7.3 the difference between the two method for larger dimensional systems will be highlighted.

7.2 Switched linear system with arbitrary switching

Consider the SLS (6), where $M = 2$ and

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.01 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -11.72 & -2 \end{bmatrix}. \qquad (46)$$

This system was studied in [3], where CPA Lyapunov functions were constructed using linear programming.

It was shown that a CPA Lyapunov function could be constructed if the triangulation $\mathcal{T}_{50}$ was used.

To construct a CPQ Lyapunov function with the proposed method, the triangulation $\mathcal{T}_4$ is used. By solving the LP problem (18) with additional linear constraints (22), a solution Φ, which parameterizes a CPQ Lyapunov function V, is found. See Fig. 4 for a visualization of V. The CPQ (LMI) method also requires a triangulation $\mathcal{T}_4$. The CPQ (LP) method from [22] requires a triangulation $\mathcal{T}_7$.

A comparison between the four methods is made in Table 1. Note the difference between the triangulations: the triangulation $\mathcal{T}_{50}$ of a two-dimensional vector space consists of 400 simplices, whereas $\mathcal{T}_4$ consists of only 32 simplices. This shows for this example that in order to construct a CPA Lyapunov function more simplices are required than to construct a CPQ Lyapunov function. The LP problem of the CPA (LP) method is solved in a similar amount of time τ_{sol} as the CPQ (LMI) method. However, due to the large number of simplices N, constructing the LP problem of the CPA (LP) method causes τ_{tot} to be much larger. The CPQ (LP) methods perform best in this example, having both the lowest τ_{sol} and τ_{tot}. This can be attributed to the low number of simplices N, which is thanks to the increased flexibility of CPQ functions over CPA functions, and the computational advantages of LP over LMI solvers.

7.3 Conewise linear systems

Consider the CLS (8), where $M = 4$,

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 1 \\ -5 & -0.1 \end{bmatrix}, \tag{47}$$

$$A_2 = A_4 = \begin{bmatrix} -0.1 & 5 \\ -1 & -0.1 \end{bmatrix} \tag{48}$$

and

$$\begin{aligned} \mathcal{F}_1 \cup \mathcal{F}_3 &= \{x \in \mathbb{R}^n \mid x_1^2 \geqslant x_2^2\} \\ \mathcal{F}_2 \cup \mathcal{F}_4 &= \{x \in \mathbb{R}^n \mid x_1^2 < x_2^2\}. \end{aligned} \tag{49}$$

This system was studied in [25], where CPQ Lyapunov functions were constructed using LMIs. In [25], a triangulation similar to $\mathcal{T}_1$ ($N = 8$) was used. However, in the example sliding mode behaviour was not taken into account, since for this particular example there are no sliding modes. If we explicitly check for the stability of sliding modes, a triangulation $\mathcal{T}_4$ ($N = 32$) is required.

To construct a CPQ Lyapunov function with LP with the proposed method, the triangulation $\mathcal{T}_4$ is used. By solving the LP problem (40) with additional linear constraints (22), a solution Φ which parameterizes a CPQ Lyapunov function V is found. See Fig. 5 for a visualization of V. The CPA (LP) method is adapted for the stability analysis of SLSs, see [37], and is applied to this example as well. The CPQ (LP) method from [22] requires the triangulation $\mathcal{T}_5$.

See Table 1 for the comparison between the three methods. Similar to the previous example, the CPA (LP) method requires the most simplices N, causing the method to have the highest τ_{tot}. The CPQ (LMI) method has the highest τ_{sol} due to requiring an LMI solver over an LP solver. The CPQ (LP) methods achieve the lowest τ_{sol} and τ_{tot} as they benefit from the best of both other methods: They require a low number of simplices N due to the flexibility of CPQ functions and they can benefit from fast LP solvers.

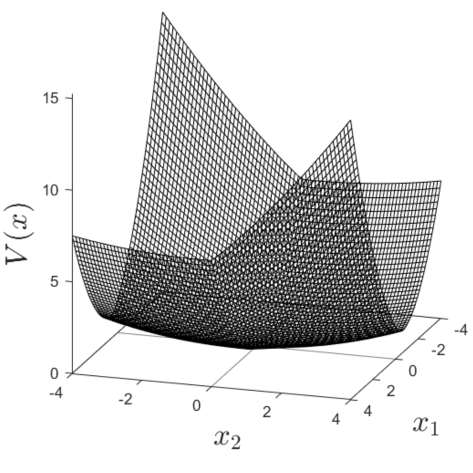

Fig. 4. CPQ Lyapunov function constructed with the proposed method for the SLS (6) with system matrices (46) and $\mathcal{T}_4$.

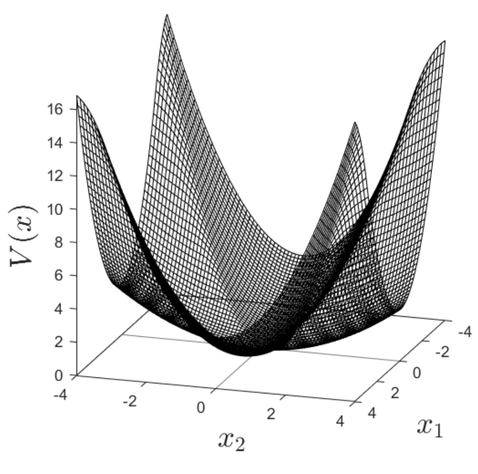

Fig. 5. CPQ Lyapunov function constructed with the proposed method for the CLS (8) with system matrices (47) and (48), subsets (49) and $\mathcal{T}_4$.

Table 2
Comparison between methods for randomly generated SLSs.

	α [%]	τ_{tot}[ms]	τ_{sol}[ms]
CPA (LP) [19]	5.10	12.8±3.91	7.81±2.75
CPQ (LP) [22]	15.9	96.9±5.57	65.4±5.16
proposed CPQ (LP)	21.4	398±26.4	148±19.3
CPQ (LMI) [26]	36.4	839±76.6	232±59.2

7.4 *Randomly generated SLSs of dimension $n = 4$*

Finally, we consider a set of 1000 GUES switched linear systems that were randomly generated. The SLSs are of the form (6) with $n = 4$ and $M = 2$. All four previously mentioned methods for constructing Lyapunov functions are applied to these systems with the triangulation $\mathcal{T}_1$. We define the accuracy α of a method as the percentage of systems for which the method can find a function V that is decreasing along all trajectories of the system (i.e., step one of the two-step procedure in Section 3.3). All four methods would succeed in finding a Lyapunov function given a sufficiently refined triangulation $\mathcal{T}_K$ (a converse result for the method in [22] is implied by our converse theorems), but we are interested in the accuracy and computation time given a set triangulation. The used code is precompiled, which reduces τ_{tot} compared to Table 1.

See Table 2 for the comparison between the four methods, where the mean and standard deviation of τ_{tot} and τ_{sol} are provided. The methods are ordered by their achieved accuracy and it can be seen that τ_{tot} and τ_{sol} increase with accuracy. The proposed method sits in between the CPQ (LP) method from [22] and the CPQ (LMI) method from [26] in both accuracy (which follows from the analysis in Sections 6.1 and 6.3) and computation time. This shows how the proposed method achieves the highest accuracy of the LP-based methods and thus comes closer to the accuracy of the CPQ (LMI) method from [26], while still having lower computation times than the LMI method.

8 Conclusion

We have presented a method for constructing CPQ Lyapunov functions for switched linear systems and conewise linear systems by making use of LP. In comparison to classical LMI methods, LP may offer computational advantages especially for problems involving a large number of optimization parameters and constraints, which is the case when accurate analysis with CPQ functions requires a partition of the state space with a large number of regions. In order to arrive at an LP problem, we formulated sufficient conditions for the copositivity of matrices in terms of testing diagonal dominance. This involves defining linear constraints that can be solved using an LP. Interestingly, despite that there is some conservatism in the formulation of copositivity of a matrix in terms of linear inequalities, we provided converse theorems guaranteeing the feasibility of our LP conditions in case of a G(U)ES SLS or CLS and a sufficiently refined partitioning. Hence, there is no conservatism in our LP conditions provided that the partitioning is sufficiently refined. The method is shown to be the least conservative of the LP-based methods [19] and [22] given a fixed partitioning and numerically competitive with the LMI-based method in [26] on various CLSs and SLSs. For future work, the method could be generalized to piecewise linear and affine systems, as well as discrete-time versions of those systems and the systems considered in this paper.

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A Auxiliary theorems

Theorem 15 *Given a matrix $B = B^\top \in \mathbb{R}^{n\times n}$, define $C = C^\top \in \mathbb{R}^{n\times n}$ as*

$$[C]_{kk} = [B]_{kk} \qquad \forall k \in \{1,2,\dots,n\} \tag{A.1}$$
$$[C]_{k\ell} = \max([B]_{k\ell},0) \quad \forall k,\ell \in \{1,2,\dots,n\}, k \neq \ell. \tag{A.2}$$

Then, C satisfies

$$\sum_{\ell=1}^{n}[C]_{k\ell} < 0 \quad \forall k \in \{1,2,\dots,n\}, \tag{A.3}$$

if and only if there exists $W = W^\top \in \mathbb{R}^{n\times n}_{\geqslant 0}$ such that $B+W$ is a negative definite and strictly diagonally dominant matrix.

PROOF. First, we will prove the "only if" part of the theorem. If C as defined in (A.1) and (A.2) satisfies (A.3), then C is negative definite (see first step in Theorem 6) and strictly diagonally dominant, because

$$\begin{aligned} 0 &> \sum_{\ell=1}^{n}[C]_{k\ell} && \Longrightarrow \\ -[C]_{kk} &> \sum_{\ell=1,\ell\neq k}^{n}[C]_{k\ell} && \Longrightarrow \\ |[C]_{kk}| &> \sum_{\ell=1,\ell\neq k}^{n}|[C]_{k\ell}|. \end{aligned}$$

Here we make use of the fact that the off-diagonal entries of C are nonnegative. Thus, $W = C - B$ is an admissible solution to ensure that $B + W$ is negative definite and strictly diagonally dominant.

Next, we will prove the "if" part of the theorem. Note that if $D := B + W$ is negative definite, its diagonal entries fulfill $[D]_{kk} = e_k^\top D e_k < 0$, where e_k is the standard k-th unit vector. Since W only contains nonnegative entries,

$$\begin{aligned} 0 > [D]_{kk} &\geqslant [C]_{kk} && \forall k \in \{1,2,\dots,n\} \\ |[D]_{k\ell}| &\geqslant |[C]_{k\ell}| && \forall k,\ell \in \{1,2,\dots,n\}, k \neq \ell. \end{aligned}$$

Therefore, C must be strictly diagonally dominant if D is strictly diagonally dominant and the diagonal of C must only contain strictly negative entries. Thus, C will satisfy (A.3). □

Theorem 16 *Given an LTI system*

$$\dot{x} = Ax, \tag{A.4}$$

where $x \in \mathbb{R}^n$ is the state and $A \in \mathbb{R}^{n\times n}$ is Hurwitz, there always exists a CPQ Lyapunov function V that satisfies the conditions of Theorem 6, provided that V is parameterized with a triangulation $\mathcal{T}$ that satisfies Assumption 14.

PROOF. First, note that (A.4) is a special case of (6) where $M = 1$ and thus Theorem 6 can be applied.

Second, suppose that the CPQ Lyapunov function V is a quadratic Lyapunov function $V = x^\top P x$, i.e., $P_i = P$ for all $i \in \{1,\dots,N\}$, where P is the solution of the Lyapunov equation

$$A^\top P + PA = -I. \tag{A.5}$$

It is a well-known result that V is then a Lyapunov function of (A.4). Now, we prove that P also satisfies the conditions of Theorem 6 to complete the proof.

The matrices Q_{ij} in (11b) will be equal to $-I$ for all $i \in \{1,\dots,N\}$ and $j = 1$. Therefore,

$$B_{ij} = R_i^\top Q_{ij} R_i = -R_i^\top R_i$$

Since the columns of R_i are vertices of a simplex $\mathcal{S}_i$ of a triangulation $\mathcal{T}$, Assumption 14 implies that B_{ij} has only negative entries on its diagonal and nonpositive entries otherwise. Then, if we define C_{ij} according to (14) and (15), the diagonal entries of C_{ij} will also be negative and the off-diagonal entries will be zero. Such matrices C_{ij} will satisfy conditions (16). □